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PRODUCTS AS REFLECTIONS

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Situations when a product of topological spaces is a reflection of a subspace are investigated. Consequences and connections: pseudocompact spaces, $\mathcal{R}$-spaces, reflections of products.

All the spaces considered are assumed to be uniformizable Hausdorff.

During the author's stay in Mathematical Center in Amsterdam, Autumn 1970, the following question was rised in a discussion: Is the complement in R^{ω_1} (R real line) of a point homeomorphic to the complement of two points? The answer is easy if one realizes a Corson's theorem [5] implying that in this case R^{ω_1} is the Hewitt realcompactification of the complements. In general, if we have a product space $\Pi\{P_i\}$ and two of its non-homeomorphic subsets A, B, we want to know whether $\Pi\{P_i\} - A$, $\Pi\{P_i\} - B$ are homeomorphic. The question is answered in the negative if one knows that any homeomorphism between $\Pi\{P_i\} - A$, $\Pi\{P_i\} - B$ extends to an autohomeo-

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morphism on $\Pi\{P_i\}$ and this last property holds in the case that $\Pi\{P_i\}$ is a reflection of both $\Pi\{P_i\}-A$, $\Pi\{P_i\}-B$. Clearly, if $\Pi\{P_i\}$ is a reflection of a space P in a reflective replete subcategory $\mathcal{K}$ of Top_H, then $P_i \in \mathcal{K}$ for all i since any reflective replete subcategory $\mathcal{K}$ of a category $\mathcal{L}$ is closed under formation of retracts in $\mathcal{L}$; so $\Pi\{P_i\}$ is a reflection of P in the smaller full subcategory of $\mathcal{K}$ generated by all the products of P_i (such a reflection will be denoted by $\beta_{\{P_i\}} P$). Thus we have got the following (trivial) assertion:

Proposition. Let X, Y be dense subspaces of $\Pi\{P_i\}$. If any continuous mapping on X or Y into P_i can be continuously extended to $\Pi\{P_i\}$, then any homeomorphism on X onto Y can be extended to an autohomeomorphism on $\Pi\{P_i\}$ and, hence, X and Y are not homeomorphic provided $\Pi\{P_i\}-X$ and $\Pi\{P_i\}-Y$ are not homeomorphic.

If we omit density of X, Y we must assume instead of it that the extensions are unique. Usually one meets dense reflections so that we shall investigate only the cases indicated in the assertion.

In the sequel we will be interested in the assumption of Proposition and shall have in mind that it entails extension of homeomorphisms and "non-homeomorphism" of complements.

By ψ, w, d, u we shall denote the following cardinal functions: pseudocharacter, weight, density, uniform character, respectively (see [14] for these and other cardinal functions on topological spaces).

A standard way how to obtain mapping-extensions from subspaces of a product is to factorize them via a subproduct where it is easier to extend the factorized mapping (e.g. [5],[10]). One usually assumes that the projection of the subspace is the whole subproduct (e.g.[3],[5]). Since this assumption will be used very often throughout the paper we will formulate it before the statements:

<u>Definition</u>. Let X be a subspace of a product $\prod\{P_i \mid i\in I\}$ and α be an infinite cardinal. The subspace X is said to have the property $V(\alpha)$ if $pr_J[X]=\prod\{P_i \mid i\in J\}$ whenever $J\subset I$, $card\, J<\alpha$.

As we shall see from the following Lemma the property $V(\alpha)$ entails inductive generation of all the projections $pr_J : X\longrightarrow \prod\{P_i \mid i\in J\}$, $card\, J<\alpha$, so that any factorization of a continuous mapping on X via pr_J is continuous.

<u>Lemma</u>. A subspace X of $\prod\{P_i \mid i\in J\}$ has $V(\alpha)$ if and only if $pr_J[\mathcal{U}\cap X]=pr_J[\mathcal{U}]$ for any canonical open set $\mathcal{U}$ in $\prod\{P_i \mid i\in I\}$ and any $J\subset I$, $card\, J<\alpha$.

<u>Proof</u>. Let $\mathcal{U}=\prod\{\mathcal{U}_i \mid i\in I\}$, $x\in pr_J[\mathcal{U}]$, $card\, J<\alpha$. In accordance with [3] we denote $R(\mathcal{U})=\{i \mid \mathcal{U}_i\neq P_i\}$; then $card\, J'<\alpha$ for $J'=J\cup R(\mathcal{U})$. Choose $x'\in pr_{J'}[\mathcal{U}]$ such that $pr_J x'=x$. $V(\alpha)$ implies the existence of an $x''\in X$ such that $pr_{J'}x''=x'$. Clearly, $x''\in\mathcal{U}\cap X$ and $pr_J x''=x$.

<u>Corollary</u>. Let X have $V(\alpha)$ in $\prod\{P_i \mid i\in I\}$ and $J\subset I$, $card\, J<\alpha$. Then the projection $pr_J : X\longrightarrow$ $\longrightarrow \{P_i \mid i\in J\}$ is open.

If X has $V(\alpha)$ in $\Pi\{P_i \mid i \in I\}$ and $f : X \rightarrow Y$ can be factorized via pr_J for a $J \subset I$, $card\, J < \alpha$, we can extend f continuously onto $\Pi\{P_i \mid i \in I\}$ into Y - e.g. [3] (the whole projection $pr_J : \Pi\{P_i \mid i \in I\} \longrightarrow$ $\rightarrow \Pi\{P_i \mid i \in J\}$ followed by the factorization). There are two general theorems about factorizations of such mappings f - for references and history see [3],[21] (a space X is said to be pseudo-α-compact [8],[13] if any locally finite open family in X is of cardinality less than α).

<u>Comfort-Negrepontis</u> [3]: Let α be a regular uncountable cardinal and X be a subspace of $\Pi\{P_i \mid i \in I\}$ with $V(\alpha)$. If X is a pseudo-α-compact, then any $f \in C(X,Y)$, Y metrizable, depends on less than α coordinates.

<u>Gleason</u> [13]: Let X be an open subset of $\Pi\{P_i \mid i \in I\}$. If all the P_i are separable, then any $f \in C(X,Y)$, Y of countable pseudocharacter, depends on countably many coordinates.

Moreover, W.W. Comfort and S. Negrepontis proved in [3] that under the assumptions stated above (α regular uncountable, X has $V(\alpha)$), X is pseudo-α-compact if and only if any finite subproduct of $\Pi\{P_i \mid i \in I\}$ is pseudo-α-compact. So this condition on X is in fact a condition on $\{P_i\}$ as in the Gleason's theorem. Only because of shorter expression we shall use the condition formulated for X. Next, we shall give slight generalizations and comments to both theorems.

It is seen from the proof of Comfort-Negrepontis' theorem [3] that one needs weaker condition on Y than that of metrizability. Define a cardinal function w in the following way: $\nu Y < \alpha$ if there is a system $\{\mathcal{U}_\xi \mid \xi < \beta\}$, $\beta < \alpha$, of reflexive relations on Y such that $\bigcap\{\mathcal{U}_\xi^{-1} \circ \mathcal{U}_\xi \circ \mathcal{U}_\xi^{-1} \circ \mathcal{U}_\xi \mid \xi < \beta\}$ is the identity and that $\mathcal{U}_\xi[x]$ is a neighborhood of x for any $x \in X$, $\xi < \beta$. Evidently, $\psi Y \leq \psi(\Delta_Y, Y \times Y) \leq \nu Y \leq u Y$ (even $\nu Y \leq u Y'$ for any coarser space Y'); $\psi(\Delta_Y, Y \times Y) = \nu Y$ provided any neighborhood of Δ_Y in $Y \times Y$ is uniformizable, i.e., if any open cover $\mathcal{A}$ has an open refinement $\mathcal{B}$ with the property: $x \in B_1, y \in B_2, B_1 \cap B_2 \neq \emptyset, B_i \in \mathcal{B} \Longrightarrow$ $\Longrightarrow (x, y) \subset A$ for an $A \in \mathcal{A}$ [1],[18]. In particular, $\psi(\Delta_Y, X \times Y) = \nu Y$ if Y is paracompact or a Σ-product of complete separable metric spaces [5] (for further cases see [17]).

Theorem 1. Let α be an uncountable regular cardinal and $X \subset \prod\{P_i \mid i \in I\}$ have $V(\alpha)$. If X is pseudo-α-compact, then any $f \in C(X, Y)$, $\nu Y < \alpha$, depends on less than α coordinates.

We shall now give the Gleason´s theorem a form similar to Theorem 1 (for α isolated and $X = \prod\{P_i \mid i \in I\}$ see [14]).

Theorem 2. Let α be an uncountable regular cardinal and $X \subset \prod\{P_i \mid i \in I\}$ have $V(\alpha)$. If $dP_i < \alpha$ for all $i \in I$, then any $f \in C(X, Y)$, $\psi Y < \alpha$ depends on less than α coordinates.

Proof. The assumption on Y implies the existence of $J_x \subset I$, $\operatorname{card} J_x < \alpha$, for any $x \in X$, such that $fx = fy$ whenever $y \in X$, $pr_{J_x} y = pr_{J_x} x$. Take now an arbitrary nonvoid $J_0 \subset I$ with $\operatorname{card} J_0 < \alpha$. Since density character of $\prod\{P_i \mid i \in J_0\}$ is less than α (α is regular), there is a set S_0' of cardinality less than α and dense in $\Pi\{P_i \mid i \in J_0\}$; take S_0 to be a subset of X such that pr_{J_0} is a bijection on S_0 onto S_0' . Now put $J_1 = J_0 \cup \bigcup\{J_x \mid x \in S_0\}$. Evidently, $\mathrm{card}\, J_1 < \alpha$ and we can construct a set S_1 for J_1 in the same way as S_0 for J_0 . By an inductive procedure we obtain sequences $\{J_n\}$ (increasing), $\{S_n\}$ such that $J_n \subset I$, $S_n \subset X$, $\mathrm{card}\, J_n < \alpha$, $\mathrm{card}\, S_n < \alpha$ and pr_{J_n} maps S_n injectively onto a dense subset of $\Pi\{P_i \mid i \in J_n\}$. Put $J = \bigcup\{J_n\}$, $S = \bigcup\{S_n\}$. First, we shall notice that $pr_J[S]$ is dense in $\Pi\{P_i \mid i \in J\}$ and that $fx = fy$ whenever $x, y \in X$, $pr_J x = pr_J y \in pr_J[S]$. Indeed, if U is a canonical open set in $\Pi\{P_i \mid i \in J\}$, then $R(U) \subset J_n$ for an n and so there is an $s \in S_n$ such that $pr_{J_n} s \in pr_{J_n}[U]$; it follows $pr_J s \in U$. To prove the second assertion, take $x, y \in X$ with $pr_J x = pr_J y \in pr_J[S]$; there is an n and $s \in S_n$ such that $pr_J s = pr_J x = pr_J y$ and, consequently, $pr_{J_s} s = pr_{J_s} x = pr_{J_s} y$ and $fs = fx = fy$ by definition of J_s . Choose now an arbitrary $x, y \in X$ such that $pr_J x = pr_J y$. For any canonical neighborhoods U, V of x, y respectively there is an $s' \in pr_J[S] \cap pr_J[U] \cap pr_J[V]$; by Lemma, there are $u_{U,V} \in U \cap X$, $v_{U,V} \in V \cap X$ such that $pr_J u_{U,V} = pr_J v_{U,V} = s'$. The nets $\{u_{U,V}\}$, $\{v_{U,V}\}$ converge in X to x, y respectively and $fu_{U,V} = fv_{U,V}$. Hence $fx = fy$. The proof is complete.

The condition " $dP_i < \alpha$ for all i " in Theorem 2 implies pseudo- α -compactness of any dense subspace in

$\Pi\{P_i \mid i \in I\}$ and hence, the condition on X in Theorem 1. On the other hand, the condition $\nu Y < \alpha$ in Theorem 1 implies the condition $\psi Y < \alpha$ in Theorem 2. In both cases the converse implications are false (take $\alpha = \omega_1$, $P_i = T_{\omega_1}$ for all i and $Y = T_{\omega_1}$). But T_{ω_1} is a bad counterexample to the "union" of both Theorems; details and generalizations will appear in a forthcoming paper.

The condition $V(\alpha)$ is essentially set-theoretic. We will show now that $V(\alpha)$ follows from "nice" topological conditions. For instance, $V(\omega_1)$ implies G_δ-density of X in $\Pi\{P_i \mid i \in I\}$ and the converse is true provided all the P_i are of countable pseudocharacters; similarly for higher cardinals.

If $pr_J[X] \neq \Pi\{P_i \mid i \in J\}$, then the complement of X contains a homeomorph of $\Pi\{P_i \mid i \in I - J\}$ as a retract. Thus, if V is a topological property preserved by retracts and such that no space $\Pi\{P_i \mid i \in I - J\}$, $\mathrm{card}\, J < \alpha$ has V but $\Pi\{P_i \mid i \in I\} - X$ has, then $V(\alpha)$ for X holds.

First take V to be a property described by cardinal functions. Let φ be a cardinal function being not increased by retracts; if $\varphi(\Pi\{P_i \mid i \in I\} - X) < \min\{\varphi(\Pi\{P_i \mid i \in I - J\}) \mid \mathrm{card}\, J < \alpha\}$, then $V(\alpha)$ holds. Since $\varphi(\Pi\{P_i \mid i \in I - J\}) \geqq \sup\{\varphi P_i \mid i \in I - J\}$ it suffices to require $\varphi(\Pi\{P_i \mid i \in I\} - X) < \min\{\sup\{\varphi P_i \mid i \in I - J\} \mid \mathrm{card}\, J < \alpha\}$ and, in particular, $\varphi(\Pi\{P_i \mid i \in I\} - X) < \min\{\varphi P_i \mid i \in I\}$.

To simplify statements we shall suppose for a moment that $P_i = P$ for all $i \in I$ and $\mathrm{card}\, P > 1$. Without loss of generality we shall also suppose $\omega_0 \leqq \alpha \leqq \mathrm{card}\, I$. The condition $\mathcal{V}(\alpha)$ (for any $\alpha \leqq \mathrm{card}\, I$) is then implied by existence of a topological property $\mathcal{V}$ preserved by retracts and such that P^I does not have $\mathcal{V}$ and $P^I - X$ has $\mathcal{V}$. The above formulas may be given in this case the following form:

$$\varphi(P^I - X) < \varphi P^I \quad \text{or} \quad \varphi(P^I - X) < \varphi P .$$

For instance, $\mathcal{V}(\alpha)$ holds if $\mathrm{card}(P^I - X) < 2^{\mathrm{card}\, I}$ or if $d(P^I - X) < dP \cdot \lg \mathrm{card}\, I$.

Theorem 3. Let P be a space and A a subspace of P^I such that $\varphi A < \varphi P^I$ for a cardinal function φ being not increased by retracts. If $\mathrm{card}\, I > \max(dP, \psi Y)$, then $P^I - A$ is $C(P^I - A, Y)$-embedded in P^I.

Proof. Put $\alpha = (\max(dP, \psi Y))^+$. Then $P^I - A$ has $\mathcal{V}(\alpha)$ and, by Theorem 2, any continuous $f: P^I - A \to Y$ can be continuously factorized via pr_J, $\mathrm{card}\, J < \alpha$ and, hence, has a continuous extension on P^I into Y.

Corollary 1. Let P be a space and A a subspace of P^I such that $\varphi A < \varphi P^I$ for a cardinal function being not increased by retracts. If $\mathrm{card}\, I > \max(dP, \psi P)$, then $\beta_P(P^I - A) = P^I$.

By a β-compact space (in the sense of Herrlich [11]) we mean a space each of its $\mathcal{Z}$-ultrafilters with β-intersection property is fixed.

Corollary 2. Let P be not a β-compact space and let $\mathrm{card}\, I > \max(dP, \psi P)$. If A is a β-compact subspace of P^I , then $\beta_P(P^I - A) = P^I$.

Proof. Put φ to be the compactness degree in the sense of Herrlich [11], i.e., φP is the first cardinal α such that P is α-compact.

Corollary 3. Let P be the space R of reals, $\operatorname{card} I > \omega_0$ and A, B non-homeomorphic subspaces of R^I with cardinalities smaller than $2^{\operatorname{card} I}$. Then $R^I - A$, $R^I - B$ are not homeomorphic.

Putting $\beta = \omega_0$ in Corollary 2 we obtain the following generalization of Theorem 1 in [22] (for a more general version see Theorems 4, 6 in the sequel):

Let P be a realcompact non-compact space and A be a compact subspace of a product P^I, where $\operatorname{card} I > dP$. Then $v(P^I - A) = P^I$.

Choose now other properties for V , e.g. to be normal or a k-space.

Theorem 4. Let P be not compact, $\alpha > wP$. Then $\beta_P(P^\alpha - A) = P^\alpha$ for any normal subspace A of P^α .

Proof. If V is the property "to be normal", then by [19],[15], P^α does not have V provided P is not compact and $\alpha > wP$. Clearly $\alpha > \max(dP, \psi P)$.

A similar assertion can be formulated for the property "to be a k-space". First we must prove the following analogon of the Noble´s theorem:

Theorem 5. Let P be not compact. Then there is α such that P^α is not a k-space.

Proof. Suppose first that P is not locally compact. Let x_0 be a point of P without compact neighborhood,

$x_1 \in P - (x_0)$ and let $\mathcal{C}$ be a system of compact sets in P containing x_0 and such that any compact set containing x_0 is contained in a member of $\mathcal{C}$. We will show that $P^{\mathcal{C}}$ is not a k-space. We may suppose that $C_0 = (x_0) \in \mathcal{C}$. For finite $F \subset \mathcal{C}$, denote $A_F = \{\{y_C\} \in P^{\mathcal{C}} \mid y_C = x_0$ for $C \in F - (C_0)$, $y_C = x_1$ for $C \in \mathcal{C} - (F \cup (C_0))$, $y_{C_0} \in X - \bigcup F\}$; $A = \bigcup\{A_F \mid F \subset \mathcal{C}, F$ finite$\}$.

(1) $\{x_0\} \in \overline{A}$: let F be a finite subset of $\mathcal{C}$, $F = (C_0, C_1, \dots, C_n)$ and U_i, $i = 0, \dots, n$, be neighborhoods of x_0 in P; then the point $\{y_C\}$, where $y_C = x_1$ for $C \in \mathcal{C} - F$, $y_{C_i} = x_0$ for $i = 1, \dots, n$, $y_{C_0} \in U_0 - \bigcup F$ (x_0 has no compact neighborhood !), belongs to $A_F \cap \cap \prod\{U_C \mid C \in \mathcal{C}\}$, where $U_C = X$ for $C \in \mathcal{C} - F$, $U_{C_i} = U_i$.

(2) K compact in $P^{\mathcal{C}}$, $\{x_0\} \in K \Longrightarrow \{x_0\} \notin \overline{K \cap A}$: there is a $C_1 \in \mathcal{C}$, $C_1 \neq C_0$ such that $C_1 \supset pr_{C_0}[K]$ and a neighborhood U of x_0 not containing x_1; assume that $\{y_C\} \in K \cap A \cap \prod\{U_C \mid C \in \mathcal{C}\}$, where $U_C = X$ for $C \in \mathcal{C} - (C_0, C_1)$, $U_{C_0} = U_{C_1} = U$. Then $\{y_C\} \in A_F$ for a finite $F \subset \mathcal{C}$; clearly $C_1 \in F$ because $y_{C_1} \neq x_1$ and, consequently, $y_{C_0} \notin C_1$ but this contradicts to $y_{C_0} \in pr_{C_0}[K] \subset C_1$.

If P is locally compact, then P^{ω_0} is not locally compact and we may accept the preceding construction for P^{ω_0} instead of P. The proof is complete.

Define γP to be the least cardinal of a base for the ideal of compact sets in P and, if P is not compact, $\gamma' P = \min\{\gamma F \mid F$ closed in P^{ω_0}, F not locally compact }. Then we have proved:

If P is not compact (i.e. $\gamma P \geq \omega_0$), then P^{α} is not a k-space provided $\alpha \geq \gamma' P$.

It is easy to show that $\gamma P \leq \gamma(P^{\omega_0}) \leq \mathrm{card}\, \exp_{\omega_0} \gamma P$ (the cardinality of the set of all countable subsets of γP, which is $2^{\omega_0} . \gamma P$ if $\mathrm{cof}\, \gamma P > \omega_0$) and $\gamma' P \leq \gamma(P^{\omega_0})$; thus, mostly, it suffices to assume $\alpha \geq 2^{\omega_0} . \gamma P$. If the continuum hypothesis does not hold, then it may happen that $\gamma(P^{\omega_0}) < 2^{\omega_0} . \gamma P$ even for non-compact spaces (e.g. $\gamma(T_{\omega_1}^{\omega_0}) = \gamma T_{\omega_1} = \omega_1$); under continuum hypothesis always $\gamma(P^{\omega_0}) = 2^{\omega_0} . \gamma P$ whenever $\mathrm{cof}\, \gamma P > \omega_0$ or $\gamma P = \omega_0$ since $\gamma(P^{\omega_0}) > \omega_0$ provided P is not compact. It is shown in [16] that if P is not countably compact, then P^{ω_1} is not a k-space. This assertion does not follow from that of ours. If continuum hypothesis is not true then it may happen that still $\gamma' N = 2^{\omega_0}$. Indeed, if F is closed in N^N and not locally compact, then F contains a closed subset homeomorphic to the space $T = N \times N \cup (\infty)$, where points of $N \times N$ are isolated and any neighborhood of ∞ contains all the $(n) \times N$, $n \in N$, except finite number. Thus $\gamma' N = \gamma T$. Since compact sets in T are of the form $(\infty) \cup A$, where $A \cap ((n) \times N)$ is finite for every n, the problem of finding γT reduces to finding small cofinal subsets in the system of all functions $f: N \longrightarrow N$, ordered pointwise. As was communicated to the author by B. Balcar and P. Štěpánek, there is an ultrafilter on N such that in the corresponding ultraproduct-model any such cofinal part is of cardinality 2^{ω_0} .

It is easy to see that Theorem 5 and remarks following it remain valid in larger classes of topological spaces than completely regular Hausdorff (of course, in dependence on definitions of local compactness and of k-spaces).

Theorem 6. Let P be not compact, $\alpha > \max(\gamma' P, dP, \psi P)$. Then $\beta_P (P^{\alpha} - A) = P^{\alpha}$ for any k-space A in P^{α} .

Corollary. Let α be an uncountable cardinal and A, B non-homeomorphic k-spaces or normal spaces in R^{α} (in particular, metrizable or compact). Then $R^{\alpha} - A$ is not homeomorphic to $R^{\alpha} - B$.

Similar statements can be given e.g. for the case when A, B are zero-dimensional subspaces of R^{α} , etc. (In general, if $\mathcal{K}$ is a productive and closed-hereditary class of spaces, i.e. epireflective in Top_H , then $P \notin \mathcal{K}, A \in \mathcal{K}$ entails $P^I - A$ has $V(\alpha)$ for all $\alpha \leq card\, I$.)

N. Noble proved in [20] that if $X \subset \Pi \{P_i \mid i \in I\}$, any finite subproduct of $\{P_i\}$ satisfies countable chain condition and any countable subproduct is perfectly normal realcompact, then $\upsilon X = \Pi \{P_i \mid i \in I\}$ if and only if X is G_δ-dense in $\Pi \{P_i \mid i \in I\}$; the conditions are satisfied if e.g. all the P_i are separable metrizable. We shall prove now a more general

Theorem 7. Let X be a subspace of a product $\Pi \{P_i \mid i \in I\}$ of realcompact spaces with countable pseudocharacters and let all the finite subproducts of $\{P_i\}$ be pseudo-ω_1-compact. Then $\upsilon X = \Pi \{P_i \mid i \in I\}$ if and only if X is G_δ-dense in $\Pi \{P_i \mid i \in I\}$.

Proof. The nontrivial part is the "if" part. If X is G_δ -dense in $\Pi\{P_i \mid i \in I\}$, then X has $V(\omega_1)$ because $\psi P_i = \omega_0$ for all i. By Comfort-Negrepontis' theorem we obtain directly that X is C-embedded in $\Pi\{P_i \mid i \in I\}$.

Corollary. Let X be a subspace of a product $\Pi\{P_i\}$ of realcompact separable spaces with countable pseudocharacters. Then $\upsilon X = \Pi\{P_i\}$ if and only if X is G_δ-dense in $\Pi\{P_i\}$.

As was remarked earlier, W.W. Comfort and S. Negrepontis proved in [3] that $X \subset \Pi\{P_i\}$ with $V(\omega_0)$ is pseudo-α-compact, α regular uncountable, if and only if any finite subproduct is pseudo-α-compact. The corresponding theorem for $\alpha = \omega_0$ is not true [2],[9] in general, but holds in special important cases - see [6],[7] for the case of compact discrete P_i. We give here a generalization of the Efimov-Engelking's theorem.

Theorem 8. Let P_i be compact for all $i \in I$ and $X \subset \Pi\{P_i \mid i \in I\}$ have $V(\omega_1)$. Then X is pseudocompact and $\upsilon X = \beta X = \Pi\{P_i \mid i \in I\}$.

Proof. It suffices to prove that X is C-embedded in $\Pi\{P_i \mid i \in I\}$ and this assertion follows directly from the Comfort-Negrepontis' theorem.

Corollary 1. Let X be a dense subspace of a product of compact metrizable spaces P_i. Then X is pseudocompact if and only if X is G_δ-dense in $\Pi\{P_i\}$.

Proof. The property "X is G_δ-dense in $\Pi\{P_i\}$" is equivalent to "X has $V(\omega_1)$" in our case. If X is pseudocompact, $J \subset I$, $\mathrm{card}\, J \leq \omega_0$, then $pr_J[X]$ is a dense pseudocompact subspace of the compact metrizable $\Pi\{P_i \mid i \in J\}$ and, hence, $pr_J[X] = \Pi\{P_i \mid i \in J\}$. The converse implication follows from Theorem 8.

Corollary 2. Let X be a dense pseudocompact subspace of a product of compact metrizable spaces P_i. Then $\upsilon X = \beta X = \Pi\{P_i\}$.

The same procedure may be used if we know that the product is pseudocompact.

Theorem 9. Let the product $\Pi\{P_i \mid i \in I\}$ be pseudocompact and $X \subset \Pi\{P_i \mid i \in I\}$ have $V(\omega_1)$. Then X is pseudocompact.

Proof. X is C-embedded in $\Pi\{P_i \mid i \in I\}$.

Corollary. If $X \subset T_{\omega_1}^{\alpha}$ has $V(\omega_1)$, then X is pseudocompact.

At the end we shall give an analogon to the Glicksberg's theorem on $\beta\Pi\{P_i\}$ [10] (it should be noted that our result is analogous to a corollary, not to the main theorem in [10]). The α-compactification of X [11] is denoted by $\beta_\alpha X$ (i.e., $X \subset \beta_\alpha X \subset \beta X$, $\beta_\alpha X$ is the set of all $\mathfrak{Z}$-ultrafilters on X with α-intersection property). First a lemma (for two factors and $\alpha = \omega_1$ see [4]).

Lemma. Let $\Pi\{P_i\}$ be C^*-embedded in $\Pi\{\beta_\alpha P_i\}$. Then $\beta_\alpha \Pi\{P_i\} = \Pi\{\beta_\alpha P_i\}$.

Proof. It is proved in [12] that α-compact spaces form the epireflective hull in Top_H of the spaces $I^\beta - (p)$, $\beta < \alpha$, where I denotes now the real interval $[0,1]$ and p is a point of I^β. Thus it suffices to prove that if $\Pi\{P_i\}$ is C^*-embedded in $\Pi\{\beta_\alpha P_i\}$, then any

$f : \Pi\{P_i\} \longrightarrow I^\beta - (p)$, $\beta < \alpha$, can be continuously extended to $\Pi\{\beta_\alpha P_i\}$ into $I^\beta - (p)$. Let $\beta < \alpha$, $f : \Pi\{P_i\} \longrightarrow I^\beta - (p)$. Then each $pr_\xi \circ f$ ($pr_\xi : I^\beta - (p) \longrightarrow I$ is the ξ's projection, $\xi < \beta$) has a continuous extension $\tilde{f}_\xi : \Pi\{\beta_\alpha P_i\} \longrightarrow I$. Put $\tilde{f} = \prod_{red}\{\tilde{f}_\xi \mid \mid \xi < \beta\}$, i.e. $\tilde{f} = \{x \longrightarrow \{\tilde{f}_\xi x\}\} : \Pi\{\beta_\alpha P_i\} \longrightarrow I^\beta$.
Suppose that $A = \tilde{f}^{-1}[p] \neq \emptyset$. We may assume that the index set of $\{P_i\}$ is the well-ordered set $\{\eta \mid \eta < \gamma\}$. Let $\{x_\eta\} \in A$ and $\{\eta_\mu\} \subset \{\eta \mid \eta < \gamma\}$ be the set of all indices η such that $x_\eta \notin P_\eta$. We shall define inductively a net $\{\{x^\mu_\eta\} \mid \mu\}$ of points of A such that $x^\mu_\eta \in P_\eta$ for all $\eta \leqq \eta_\mu$, $x^\mu_\eta = x_\eta$ for $\eta \notin \{\eta_\mu\}$ or $\eta > \eta_\mu$ and such that $x^{\mu'}_\eta = x^{\mu''}_\eta$ for $\mu' < \mu''$, $\eta \leqq \eta_{\mu'}$. For $\mu = 0$ there is a $x^0_{\eta_0} \in P_{\eta_0}$ such that $\{x^0_\eta\} \in A$, where $x^0_\eta = x_\eta$ for $\eta \neq \eta_0$, for otherwise $\emptyset \neq \widetilde{\beta_\alpha P_{\eta_0}} \cap A \subset \widetilde{\beta_\alpha P_{\eta_0}} - \tilde{P}_{\eta_0}$ (here $\widetilde{\beta_\alpha P_{\eta_0}}$ is the copy of $\beta_\alpha P_{\eta_0}$ by embedding $\{x \rightarrow \{x_\eta\} \mid \mid x_{\eta_0} = x,\ x_\eta = x_\eta$ for $\eta \neq \eta_0\}$ and similarly $\tilde{P}_{\eta_0}$), which is impossible because, in that case, $f / \tilde{P}_{\eta_0} : \tilde{P}_{\eta_0} \rightarrow$ $\rightarrow I^\beta - (p)$ and has the unique continuous extension $\tilde{f} / \widetilde{\beta_\alpha P_{\eta_0}} : \widetilde{\beta_\alpha P_{\eta_0}} \longrightarrow I^\beta - (p)$. Suppose that $\{x^\mu_\eta\}$ are defined for all $\mu < \tilde{\mu}$ and put $t = \{t_\eta\} = \lim\{\{x^\mu_\eta\} \mid \mid \mu < \tilde{\mu}\}$ (the net $\{\{x^\mu_\eta\} \mid \mu < \tilde{\mu}\}$ is coordinate-wise almost constant). Then $t \in A$ and we can construct $\{x^{\tilde{\mu}}_\eta\}$ from $t, \eta_{\tilde{\mu}}$ in the same way as $\{x^0_\eta\}$ from $\{x_\eta\}, \eta_0$; $\{x^{\tilde{\mu}}_\eta\}$ has the required properties.

The limit $\lim\{\{x^\mu_\eta\} \mid \mu\}$ is a point of

$A \cap \Pi\{P_i\}$ - a contradiction.

Theorem 10. Let $\Pi\{P_i \mid i \in I\}$ be pseudo- ω_1 -compact. Then $\beta_\alpha \Pi\{P_i \mid i \in I\} = \Pi\{\beta_\alpha P_i \mid i \in I\}$ if and only if $\beta_\alpha \Pi\{P_i \mid i \in J\} = \Pi\{\beta_\alpha P_i \mid i \in J\}$ for all $J \subset I$, $\mathrm{card}\, J < \omega_1$.

Proof. One implication is clear. To prove the other, by the preceding Lemma, it suffices to show that $\Pi\{P_i \mid i \in I\}$ is C^*-embedded in $\Pi\{\beta_\alpha P_i \mid i \in I\}$. If $f : \Pi\{P_i \mid i \in I\} \longrightarrow [0,1]$ then, by the Comfort-Negrepontis' theorem, $f = f' \circ pr_J$ for a $J \subset I$, $\mathrm{card}\, J \leq \omega_0$; by our condition f' can be extended on $\Pi\{\beta_\alpha P_i \mid i \in J\}$ and, hence, f can be extended to $\Pi\{\beta_\alpha P_i \mid i \in I\}$.

It may be interesting that under pseudo- ω_1 -compactness of the product, the commutation of Π with β_α depends on countable subproducts for any α . In the Glicksberg's theorem, i.e. $\alpha = \omega_0$, there is no loss of generality in assumption that the product is pseudo- ω_1 -compact but, clearly, this is not the case for $\alpha > \omega_0$. The only generalization of Theorem 10 we know is to put any uncountable regular α instead of ω_1 .

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