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ON A CONNECTION WITH TORSION ZERO [x)]

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1. The existence of a connection without torsion on the quotient bundle of a distribution on a manifold is equivalent to the integrability of that distribution. More precisely, suppose that M is a C^∞ -manifold (all maps and all objects will be C^∞ throughout this paper) and E a subbundle of the tangent bundle T of M. The distribution E is integrable, i.e. E is tangent to the leaves of a foliation on M, if and only if there exists a linear connection on $Q = T/E$ which has zero torsion. This note contains an attempt to give some algebraic criterion for the existence of a torsionless connection on Q for a given distribution E. The main result, necessary and sufficient conditions for the existence of a torsionless connection, is stated in terms of a twisting cochain ϕ from the coalgebra $\beta\bar{V}$, associated with the Weil algebra [2], to the exterior algebra E of differential forms on the principal bundle P associated with Q,

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and its affine extension.

The existence of a torsionless connection on $Q = T/E$ in relation to the integrability of E will be discussed elsewhere.

An analogous criterion for the integrability of a complex analytic bundle $E \longrightarrow T(M)$ over a complex analytic manifold M can be given in the same way.

Furthermore, let us assume that M is a $2n$-dimensional manifold with an almost complex structure on it. The existence of a torsionless connection on the almost complex principle bundle $\pi : P \longrightarrow M$, with the structure group $GL(n, C)$ is a necessary and sufficient condition for the almost complex structure to be integrable. Therefore an analogue of the condition mentioned above would be relevant for the existence of integrable almost complex structure.

2. Let $\pi : P \longrightarrow M$ be a principal bundle with the structure group $G = GL(q, R)$. Each element x of the Lie algebra $\underline{G}$ of G defines an associated fundamental vector field, denoted also by x, on P. It is the vector field tangent to the orbits of the right action of the one-parametric subgroup $\{exp\ tx\}$ of G on P. Then on the exterior algebra of differential forms $E = \bigoplus_{r \geq 0} E^r$ on P there are well defined operations:
d - the differential (antiderivation of degree $+1$),
$i(x)$ - the interior product (antiderivation of degree -1) for $x \in \underline{G}$, L_x - the derivative with respect to x (derivation of degree 0).

The Weil algebra of the Lie algebra $\underline{G}$ is the tensor product $W = S \otimes A$ of the symmetric algebra S and of the exterior algebra A of the dual G^* of $\underline{G}$. It is a differential graded algebra with antiderivation δ of degree $+1$, and antiderivation $i(x)$ of degree -1, and derivation L_x of degree 0; for all $x \in G$. Let us identify S^1 with A^1 via the obvious isomorphism $h : A^1 \longrightarrow S^1$, and for $x' \in A^1$ denote $\tilde{x}' = h(x')$.

Definition 2.1. A linear connection on the principal bundle $\pi : P \longrightarrow M$ is a linear mapping

$$(2.1) \qquad f : A^1 \longrightarrow E^1$$

such that for any $x \in G$ and $x' \in G^* = A^1$ holds:

(i) $i(x) \cdot f(x') = i(x) \cdot x'$,

$$(2.2) \qquad \text{(ii) } L_x \cdot f(x') = f(L_x(x')) .$$

The curvature of the connection f is the mapping

$$(2.3) \qquad \tilde{f} : S^1 \longrightarrow E^2$$

given by the formula

$$(2.4) \qquad \tilde{f}(\tilde{x}') = d(f(x')) - f(\delta x') .$$

Let us denote by P_+ the affine extension of P and by f_A the affine connection (linear connection on P_+), associated with f. Its curvature will be denoted by $\tilde{f}_A$. And G_+, $\underline{G}_+$, S_+, A_+, E_+ stand for the corresponding objects for the affine extension. The operators i, L, δ can be extended to the affine extension in an obvious way.

A linear connection on P can also be defined by an 1-form ω on P with values in G with certain conditions, analogous to (2.2), satisfied.

The affine connection associated with the linear connection (given by the 1-form ω) is given by the 1-form ω_+ such that the diagram

$$(2.5) \qquad \begin{array}{ccc} T(P_+) & \xrightarrow{\omega_+} & G_+ \\ \iota \big\downarrow & & \big\downarrow \iota \\ T(P) & \xrightarrow{\omega} & G \end{array}$$

commutes. And the fact that ω_+ is associated with ω is expressed in the following way:

$$(2.6) \qquad \iota^* \omega_+ = \omega + \theta \quad ,$$

where θ is a 1-form on $\pi : P \longrightarrow M$ with values in F (considering the semidirect product $G_+ = G + F$), which is zero on vertical vectors, and with respect to the right action on P

$$(2.7) \qquad R_g^* \, \theta = g^{-1} \theta \qquad \text{for any } g \in G \ .$$

If we denote by D the covariant differential of the connection ω , we get the curvature Ω and torsion Θ forms on P by the formulas

$$(2.8) \qquad \Omega = D\omega \ , \quad \Theta = D\theta \quad .$$

The mapping $\iota^* \omega_+ : T(P) \longrightarrow G_+$ in its dual form gives

Proposition 2.1. A linear connection f together with a 1-form θ induce a linear mapping

(2.9) $$f_+ : A^1_+ \longrightarrow E^1$$

such that for any $x \in G$ and $x'_+ \in A^1_+$

(2.10) (i) $$i(x) \cdot f_+(x'_+) = i(x) \cdot x'_+ \quad ,$$

(ii) $$L_x \cdot f_+(x'_+) = f_+(L_x(x'_+))$$

and the mapping

(2.11) $$\tilde{f}_+ : A^1_+ \longrightarrow E^2$$

defined by the formula

(2.12) $$\tilde{f}_+(x'_+) = d(f_+(x'_+)) - f_+(\delta x'_+)$$

is the composition

(2.13) $$\tilde{f}_+ = \tilde{f} + \tilde{f}_0$$

of the curvature and torsion of the connection f .

Proof. The properties (i),(ii) follow immediately from the definition of a connection. It is enough to notice that f_+ is a composition of a connection f and a linear mapping $f_0 : A^1_0 \rightarrow E^1$ $(0 \leftarrow A'_0 \leftarrow A'_+ \leftarrow A' \leftarrow 0)$ which is dual to the 1-form θ .

And the decomposition of $\tilde{f}_+$ is easily seen by dualizing the situation. Let ξ_1 , ξ_2 be two vector fields on P . Then for any

$x'_+ = x' + x'_0 \in A'_+ = A' \oplus A'_0$, $\langle d(f_+(x'_+)) - f_+(\delta x'_+), \xi_1 \wedge \xi_2 \rangle =$

$$\langle d(f+f_0)(x'+x_0') - (f+f_0)(\sigma(x'+x_0')), \xi_1 \wedge \xi_2 \rangle =$$
$$= \langle df(x') + df_0(x_0') - f(\sigma x') - f(\sigma x_0') - f_0(\sigma x_0'), \xi_1 \wedge \xi_2 \rangle =$$
$$= \langle df(x') - f(\sigma x'), \xi_1 \wedge \xi_2 \rangle +$$
$$+ \langle df_0(x_0') - f(\sigma x_0') - f_0(\sigma x_0'), \xi_1 \wedge \xi_2 \rangle =$$
$$= \langle x', (d\omega + [\omega,\omega])(\xi_1, \xi_2)\rangle + \langle x_0', d\theta(\xi_1, \xi_2)\rangle -$$
$$- \langle x_0, [\theta,\omega](\xi_1, \xi_2)\rangle =$$
$$= \langle x', \Omega(\xi_1, \xi_2)\rangle + \langle x_0', \Theta(\xi_1, \xi_2)\rangle =$$
$$= \langle x_+', (\Omega + \Theta)(\xi_1, \xi_2)\rangle \quad .$$

3. As was remarked in [2], the notion of a linear connection makes sense if the exterior algebra E is a graded differential algebra with an action of the group G on it. We shall adopt this more algebraic point of view.

Let S^p be the p-th symmetric product of G^* and A^q the q-th exterior product of G^*. Then we define

$$V^i = \bigoplus_{\substack{p+q=i \\ p\neq 0}} (S^p \otimes A^q) \text{ for } i > 1,\ V^1 = 0,\ V^0 = R \ .$$

The vector space $V = \bigoplus_{i\geq 0} V^i$ is the cochain complex over R with the antiderivation σ (this operator is defined on the Weil algebra in [2]; it is extended to V^0 by the requirement $\sigma(V^0) = 0$) . Let $\alpha : R \to V^0$ be the augmentation isomorphism, and

$$\overline{V} = \bigoplus_{i \geq 0} \overline{V}^i \ , \quad \overline{V}^i = V^i \qquad \text{for } i > 0 , \ \overline{V}^0 = 0 ,$$

be the reduced cochain complex. There is a natural DGA $\mathbb{R}$ -algebra structure on V with the multiplication

$$(\sigma_1 \otimes \alpha_1) \cdot (\sigma_2 \otimes \alpha_2) = \sigma_1 \cdot \sigma_2 \otimes \alpha_1 \cdot \alpha_2 \qquad \text{for}$$

any $\sigma_1 \otimes \alpha_1 \in S^{p_1} \otimes A^{q_1}$, $\sigma_2 \otimes \alpha_2 \in S^{p_2} \otimes A^{q_2}$,

where $\sigma_1 \cdot \sigma_2$ is the symmetric and $\alpha_1 \cdot \alpha_2$ the exterior product. Now we associate with V the DGA-coalgebra $\beta\overline{V}$. First define $\beta\overline{V} = \mathbb{R} \oplus \overline{V} \oplus (\overline{V} \otimes \overline{V}) \oplus \ldots$. Denote by $[v_1, \ldots, v_n]$ an element $v_1 \otimes \ldots \otimes v_n \in \overline{V} \otimes \ldots \otimes \overline{V}$ (n -times). For an element $v \in \overline{V} = \bigoplus_{p+q=i\geq 1} (S^p \otimes A^q)$, $v = \bigoplus_{p+q=i\geq 1} v_{p,q}$ let us define $\overline{dim}\, v = \bigoplus_{p+q=i\geq 1} \overline{dim}\, v_{p,q}$,and $\overline{dim}\, v_{p,q} = 2p + q$. And finally for $[v_1, \ldots, v_n]$ define $dim\,[v_1, \ldots, v_n] = 1 - n + \overline{dim}\, v_1 + \ldots + \overline{dim}\, v_n$. This gives a gradation on the module $\beta\overline{V} = \bigoplus_{i \geq 0} (\beta\overline{V})^i$.

In low dimensions we get for example

$$(\beta\overline{V})^0 = \mathbb{R} \ ; \ (\beta\overline{V})^1 = 0 \ , \ (\beta\overline{V})^2 = S^1 \otimes 1 \ ,$$

(3.1)

$$(\beta\overline{V})^3 = (S^1 \otimes 1) \otimes (S^1 \otimes 1) \oplus (S^1 \otimes A^1) \ , \text{ etc} \ .$$

The diagonal map

$$\nabla : \beta\overline{V} \rightarrow \beta\overline{V} \otimes \beta\overline{V}$$

is defined in the usual way

$$\nabla [v_1, \dots, v_t] = \sum_{i=0}^{n} [v_1, \dots, v_n] \otimes [v_{i+1}, \dots, v_n] ,$$

and for $c \in \operatorname{coker} \alpha$, $\overline{\nabla}(c) = \nabla(c) - 1 \otimes c - c \otimes 1$.

Using the antiderivation δ , defined on the Weil algebra, and the algebra structure on V we can define an antiderivation ∂ of degree $+1$ on $\beta \overline{V}$ by the formula

$$\partial [v_1, \dots, v_n] = \sum_{i=1}^{n} (-1)^{\overline{\dim}\, v_1 + \dots + \overline{\dim}\, v_{i-1}} [v_1, \dots, \delta v_i, \dots, v_n] +$$

$$+ \sum_{i=1}^{n} (-1)^{\overline{\dim}\, v_1 + \dots + \overline{\dim}\, v_{i-1}} [v_1, \dots, v_i\, v_{i+1}, \dots, v_n] .$$

Now we are in a position to state

Proposition 3.1. $\beta \overline{V}$ is a DGA coalgebra with the coproduct $\overline{\nabla}$, grading $\dim$ and differentiation ∂ .

The mapping $\pi : \beta \overline{V} \longrightarrow \overline{V}$ such that $\pi | \overline{V}$ = identity, $\pi : \overline{V} \otimes \dots \otimes \overline{V}$ (n-times, $n \geqq 2$) $\longrightarrow 0$, $\pi(\mathbb{R}) = 0$ is a twisting cochain, i.e. in particular

$$\pi\, \partial [v_1, \dots, v_n] = \partial \pi [v_1, \dots, v_n] + \mu (\pi \otimes \pi) \overline{\nabla} [v_1, \dots, v_n]$$

(μ stands for the product in $\overline{V}$), $[v_1, \dots, v_n] \in \overline{V} \otimes \dots \otimes \overline{V}$ (n -times).

Proof. Is rather straightforward by induction.

Let us denote by $\tilde{f}^i$ the i-th symmetric product of $\tilde{f}$ and by f^k the k-th exterior product of f . Then define a linear map (a chain)

$$\phi_k : (\beta \overline{V})^k \longrightarrow E^k , \quad k \geqq 0 ,$$

$$\phi_k : (\beta \overline{V})^k | \underbrace{\overline{V} \otimes \dots \otimes \overline{V}}_{n\text{-times}} \longrightarrow 0 \text{ for } n \geqq 2, \ \phi_k(\mathbb{R}) = 0, \ k \geqq 0$$

by the formula

(3.2) $$\phi_n = \bigoplus_{2i+j=n} (\tilde{f}^i \otimes f^j) \quad (\text{on } \bar{V}) .$$

Lemma 3.1. Let $E = \bigoplus_{n \geq 0} E^n$ be a DGA algebra, $\beta\bar{V} = \bigoplus_{n \geq 0} (\beta\bar{V})^n$ the DGA coalgebra associated to the Weil algebra, and f a linear connection. Then the chain $\phi = \sum \phi_n \in C^*(\beta\bar{V}, E)$ is a twisting cochain, i.e.

(i) $\phi_n \in C^n(\beta\bar{V}, E)$, $\phi_1 = 0$, $\phi((\beta\bar{V})^k) \subset E^k$,

(3.3)

(ii) $\phi_0 \cdot \alpha = 0$ and $d\phi_{n-1} = \phi_n \partial - \mu\left(\sum_{j=2}^{n-2} \phi_j \otimes \phi_{n-j}\right)\bar{V}$.

Proof. The not so obvious part of the lemma is the formula (ii). This is proved by induction. For $n = 2$ we have the diagram

(3.4) $$\begin{array}{ccc} (\beta\bar{V})^2 & \xrightarrow{\phi^2} & E^2 \\ \partial \downarrow & & \downarrow d \\ (\beta\bar{V})^3 & \xrightarrow{\phi^3} & E^3 \end{array}$$

where $(\beta\bar{V})^2$, $(\beta\bar{V})^3$ are given by (3.1). For any $[v] \in (\beta\bar{V})^2$ we have the formula $\phi^3\partial([v]) = \phi^3[\sigma v] = d\phi^2([v])$. Therefore $d\phi^2 = \phi^3\partial$. For $n = 3$ we have to consider the diagram

$$\begin{array}{ccc} (\beta\bar{V})^3 & \xrightarrow{\phi^3} & E^3 \\ \partial \downarrow & & \downarrow d \\ (\beta\bar{V})^4 & \xrightarrow{\phi^4} & E^4 \end{array}$$

where $(\beta\bar{V})^3$ is given by (3.1) and

$$(\beta\bar{V})^4 = (S^1\otimes 1)\otimes(S^1\otimes 1)\otimes(S^1\otimes 1)\oplus(S^1\otimes A^1)\otimes(S^1\otimes 1)\oplus$$

$$\oplus(S^1\otimes 1)\otimes(S^1\otimes A^1)\oplus(S^2\otimes 1)\oplus(S^1\otimes A^2)\ .$$

For $[v_1, v_2] + [v_3] \in (\beta\bar{V})^3$ we get, by the definition, $\phi^3([v_1,v_2]+[v_3]) = \phi^3([v^3])$; and from [2] follows that

$$d\,\phi^3([v_3]) = \phi^4([\delta v^3])\ .$$

But

$$\partial([v_1,v_2]+[v_3]) = [\delta v_1, v_2] + (-1)^{\overline{\dim v_1}}[v_1,\delta v_2] +$$
$$+ [v_1 \cdot v_2] + [\delta v_3]\ ,$$

and

$$\phi^4\partial([v_1,v_2]+[v_3]) = \phi^4([v_1\cdot v_2]+[\delta v_3])\ .$$

This shows that

$$d\phi^3([v_1,v_2]+[v_3]) = \phi^4\partial([v_1,v_2]+[v_3]) - \phi^4([v_1\cdot v_2])\ .$$

On the other hand

$$\bar\nabla([v_1,v_2]+[v_3]) = [v_1]\otimes[v_2]\ ,$$

$$\mu(\phi^2\otimes\phi^2)([v_1]\otimes[v_2]) = \mu((\tilde{f}\otimes 1)\otimes(\tilde{f}\otimes 1))([v_1]\otimes[v_2]) =$$

$$= (\tilde{f}^2\otimes 1)([v_1\cdot v_2]) = \phi^4([v_1\cdot v_2])\ .$$

Therefore

$$d\,\phi^3 = \phi^4\partial - \mu(\phi^2\otimes\phi^2)\bar\nabla\ .$$

Now let as assume that (ii) holds. We want to show that

$$d\,\phi_k = \phi_{k+1}\,\partial - \mu\Big(\sum_{j=2}^{k-1}\phi_j\otimes\phi_{k+1-j}\Big)\bar\nabla\ .$$

We have to consider the diagram

$$\begin{array}{ccc} (\beta\bar{V})^k & \xrightarrow{\ \phi^k\ } & E^k \\ \partial\downarrow & & \downarrow d \\ (\beta\bar{V})^{k+1} & \xrightarrow{\ \phi^{k+1}\ } & E^{k+1} \end{array}$$

with

$$(\beta\overline{V})^{k} = \Big(\bigoplus_{2i+j=k} (S^{i}\otimes A^{j})\Big) \oplus \Big(\bigoplus_{2p+2u+q+v=k+1} (S^{p}\otimes A^{q})\otimes(S^{u}\otimes A^{v})\Big).$$

Let us take an element $[u]+[v_1,v_2] \in (\beta\overline{V})^{k}$. By the definition we have $d\phi^{k}([u]+[v_1,v_2]) = d\phi^{k}([u])$. We know from [2] that this is equal to $\phi^{k+1}\partial([u]) = \phi^{k+1}([\delta u])$. This shows that

$$d\phi^{k}([u]+[v^1,v^2]) = \phi^{k+1}\partial([u]+[v^1,v^2]) - \phi^{k+1}\partial([v^1,v^2]) .$$

Now we have to look more carefully on the term $\phi^{k+1}\partial([v_1, v_2])$. We want to show that

$$\begin{array}{ccc}
\bigoplus_{2p+2u+q+v=k+1} (S^{p}\otimes A^{q})\otimes(S^{u}\otimes A^{v}) & \xrightarrow{\ \overline{V}\ } & (\beta\overline{V})\otimes(\beta\overline{V}) \\
\Big\downarrow \partial & & \Big\downarrow \hat{\mu} \\
(\beta\overline{V})^{k+1} & \xrightarrow{\ \phi^{k+1}\ } & E^{k+1}
\end{array}$$

$\hat{\mu} = \mu \sum_{j=2}^{k-1} (\phi_j \otimes \phi_{k+1-j})$, is a commutative diagram.

Because $\overline{V}([v_1, v_2]) = [v_1] \otimes [v_2]$, we get this easily from the definition. Namely

$$\mu \sum_{j=2}^{k-1} (\phi^{j} \otimes \phi^{k+1-j})([v_1]\otimes[v_2]) = \phi_{k+1}([v_1 \cdot v_2]) .$$

This finishes the proof.

<u>Lemma 3.2.</u> Let $E = \bigoplus_{k\geq 0} E^{k}$ be a DGA algebra, $\beta\overline{V}_{+} = \bigoplus_{k\geq 0} (\beta\overline{V}_{+})^{k}$ the DGA coalgebra associated with the affine extension of the Weil algebra (G^* is taken as the basic element instead of G^*), and f_{+} be the linear map

given by (2.9).

Then the chain $\phi_+ = \Sigma\, \phi_+^r \in C^*(\beta\overline{V}_+ , E)$, defined by the formula

$$\phi_+^r = \bigoplus_{2i+j=r} (\tilde{f}_+^i \otimes f_+^j)$$

is a twisting cochain, i.e.

(i) $\phi_+^r \in C^r(\beta\overline{V}_+, E)$, $\phi_+^1 = 0$, $\phi_+((\beta\overline{V}_+)^k) \subset E^k$,

(ii) $\phi_+^0 \cdot \alpha = 0$ and $d\phi_+^{r-1} = \phi_+^r \partial - \mu(\sum_{j=2}^{r-2} \phi_+^j \otimes \phi_+^{r-j})\overline{V}$.

<u>Proof</u>. Follows the lines of the proof of Lemma 3.1.

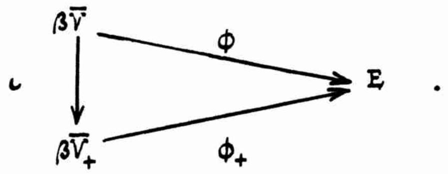
As G_+ is the affine extension of the linear group $G = GL(q, R)$, it has a normal subgroup G_0 of translations. So that there is an obvious exact sequence

$$0 \longrightarrow G_0 \longrightarrow G_+ \longrightarrow G \longrightarrow 0 .$$

And for the duals of the corresponding Lie algebras, the exact sequence

$$0 \longrightarrow G^* \xrightarrow{e} G_+^* \longrightarrow G_0^* \longrightarrow 0 .$$

The injection ι gives the injective map $\iota : \beta\overline{V} \longrightarrow \beta\overline{V}_+$, and we have the commutative diagram



Now ϕ_+ can be looked at as an extension of ϕ . And the following is immediate

<u>Theorem 3.1</u>. A connection f has zero torsion if and only if the twisting cochain ϕ_+ is the trivial extension of ϕ .

R e f e r e n c e s

[1] BROWN,Jr.,E.H.: Twisted tensor products I, Ann.of Math.69(1959),223-246.

[2] CARTAN H.: Séminaire notes, 1949-50.

[3] HAEFLIGER A.: Feuilletages sur les variétés ouvertes, Topology 9(1970),183-194.

[4] NOMIZU K.: Lie groups and differential geometry,1956.

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