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ON A CERTAIN SUM IN NUMBER THEORY III.

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§ 1. Introduction

Let r be a positive integer and let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be given real numbers. Let, for a positive integer k,

$$P_k = \max_{j=1,2,\ldots,r} \langle \alpha_j k \rangle ,$$

where $\langle t \rangle$, for a real t, denotes the distance of t from the nearest integer.

Many papers in the theory of numbers are devoted to the investigation of different sums, which contain the expression P_k. Let us recall, for example, the papers [2] and [3]. In these papers the investigation was usually restricted to the case $r = 1$. In the previous papers (see [4] and [5]) the sum

$$F(x) = \sum_{k \le \sqrt{x}} k^{\varphi} \min{}^{\beta} \left(\frac{\sqrt{x}}{k}, \frac{1}{P_k} \right)$$

was considered. Here φ and β are non-negative real numbers and we put $\min(A, \frac{1}{B}) = A$ for $B = 0$. Using Lemma 1

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(see below), which was first proved in the recent paper [1], it has been proved, among other results, that

$$\limsup_{x \to +\infty} \frac{\lg F(x)}{\lg x} = \max\left(\frac{\beta\gamma + \varphi}{2(\gamma+1)}, \frac{\varphi+1}{2}\right).$$

Here, γ is the least upper bound of all the numbers $\tau > 0$ for which the inequality

$$P_{\kappa} \leq \kappa^{-\tau}$$

has infinitely many solutions in positive integers κ.[1] (For $\gamma = +\infty$ we put $\frac{\beta\gamma + \varphi}{2(\gamma+1)} = \frac{\beta}{2}$.)

This result, together with other results of the present author yields the solution of the basic problem in the theory of lattice points with weight in rational, high-dimensional ellipsoids (see [5], Theorems 3 and 4).

Let $Q(\underset{\sim}{u})$ be a positive definite quadratic form in r variables with a symmetric integral coefficient matrix and determinant D. Let us put, for $x > 0$,

$$P(x) = \sum e^{2\pi i \sum_{j=1}^{r} \alpha_j u_j} - \frac{\pi^{\frac{r}{2}} x^{\frac{r}{2}} e^{2\pi i \sum_{j=1}^{r} \alpha_j}}{\sqrt{D}\,\Gamma(\frac{r}{2}+1)} \delta ,$$

where $\delta = 1$ if all the α_j are integers, and $\delta = 0$ otherwise. Here the summation runs over all r-triples $u = (u_1, u_2, \ldots, u_r)$ of integers such that $Q(u) \leq x$. Then

$$\limsup_{x \to +\infty} \frac{\lg |P(x)|}{\lg x} = \left(\frac{r}{4} - \frac{1}{2}\right) \frac{2\gamma+1}{\gamma+1},$$

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1) In the sequel we denote this value by $\gamma(\alpha_1, \alpha_2, \ldots, \alpha_r)$.

provided $\frac{1}{\gamma} \leq \frac{k}{2} - 2$, where $\gamma = \gamma(\alpha_1, \alpha_2, \dots, \alpha_k)$.
(For $\gamma = +\infty$ we put $\frac{1}{\gamma} = 0$, $\frac{2\gamma+1}{\gamma+1} = 2$.)

The aim of this paper is to investigate other sums by similar methods. The results about the function $G(x)$ (defined below) generalize the results of papers [2] and [3]. The results about the function $H(x)$ (also defined below) play the essential role in obtaining Ω-estimates of the "lattice remainder term" in the theory of lattice points in high-dimensional spheres with an arbitrary center, i.e., the function

$$P(x) = \sum 1 - \frac{\pi^{\frac{k}{2}} x^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+1)} ,$$

where the summation runs over all k-triples $u = (u_1, u_2, \dots, u_k)$ of integers such that

$$(u_1 + b_1)^2 + (u_2 + b_2)^2 + \dots + (u_k + b_k)^2 \leq x .$$

Here, $b_1, b_2, \dots, b_k$ are given real numbers and $x > 0$. We announce here the basic result (for the proof see [6]):

$$\limsup_{x \to +\infty} \frac{\lg |P(x)|}{\lg x} = \frac{k}{2} - 1 - \frac{1}{2\gamma} ,$$

where $\gamma = \gamma(b_1, b_2, \dots, b_k)$, provided $k \geq 4 + \frac{2}{\gamma}$ (for $\gamma = +\infty$ we put $\frac{1}{\gamma} = \frac{1}{2\gamma} = 0$).

In the sequel, we let the letter c denote (generally different) constants depending only on α_j, φ, β

and γ. We write $A << B$ instead of $|A| \leq cB$; if, in addition, $B << A$, we write $A \asymp B$. h, k, ℓ and n mean non-negative integers, $h > 0$, $k > 0$. Let us define the symbol $B^{\{\tau\}}$, for positive B and real τ as follows:

$$B^{\{\tau\}} = \begin{cases} \dfrac{B^{\tau}}{\tau} & \text{for } \tau > 0 , \\ \lg B & \text{for } \tau = 0 , \\ 1 & \text{for } \tau < 0 . \end{cases}$$

The starting point of our consideration is the following simple lemma which we mentioned above.

Lemma 1. Let ℓ and M be integers, $M > 0$ and let γ be a positive real number. Let the inequality

$$P_k >> k^{-\gamma} \tag{1}$$

hold for all k. Then there are at most

$$c\,2^{-\frac{\ell}{\gamma}} M$$

numbers k such that $M \leq k \leq 2M$ and

$$2^{-\ell-1} \leq P_k < 2^{-\ell} . \tag{2}$$

Proof. Let $M \leq k_1 < k_2 < \dots < k_\nu \leq 2M$ be positive integers fulfilling the inequality (1). Denote by X the smallest k such that $P_k < 2 . 2^{-\ell}$. From the obvious inequality $\langle \xi_1 \pm \xi_2 \rangle \leq \langle \xi_1 \rangle + \langle \xi_2 \rangle$, for ξ_1 and ξ_2 real, we obtain

$$k_1 \geq X,\ k_2 - k_1 \geq X, \dots,\ k_\nu - k_{\nu-1} \geq X$$

and then $k_\nu \geq \nu X$. Hence by assumption (1) we have

$$2 \cdot 2^{-\ell} > P_{k} >> K^{-\gamma} \geq \left(\frac{\nu}{k_\nu}\right)^{\gamma} \geq \left(\frac{\nu}{2M}\right)^{\gamma} ,$$

and we conclude that

$$\nu << 2^{-\frac{\ell}{\gamma}} M .$$

From this lemma we obtain immediately:

Lemma 2. Let ℓ, M, γ be as in Lemma 1. Then there is a constant $c_1 = c$ such that

$$P_{k} \geq 2^{-\ell}, \quad k = M, M+1, \dots, 2M ,$$

provided $2^{\ell} \geq c_1 M^{\gamma}$.

§ 2. The sum $G(x)$

Let $P_k > 0$ for all k, i.e., at least one of the numbers $\alpha_1, \alpha_2, \dots, \alpha_k$ is irrational. Let φ, β and x be real numbers, $x > c$. We consider the sum

(3) $$G(x) = \sum_{k \leq x} k^{\varphi} P_k^{-\beta} .$$

Obviously

$$G(x) \geq \sum_{k \leq x} k^{\varphi} ,$$

provided $\beta \geq 0$. From Lemma 1 we see immediately that there are constants $c_1 = c$ and $c_2 = c$ such that the inequality $P_k \geq c_1$ is fulfilled for at least $c_2 x$ values of $k \leq x$. Thus, the relation

$$G(x) >> \sum_{k << x} k^{\varphi}$$

holds for any β , i.e.

(4) $$G(x) >> x^{\{\varphi+1\}} .$$

Let $\beta \geq 0$ and let us suppose that the inequality

(5) $$P_{k} << k^{-\gamma}$$

is fulfilled for infinitely many k, say $k = k_n$, $n = 1, 2, \dots$, where $\gamma > 0$. Then $G(k_n) >> k_n^{\varphi+\beta\gamma}$, $n = 1, 2, \dots$. In other words

(6) $$G(x) = \Omega(x^{\varphi+\beta\gamma}) .$$

Now, we pass to the O-estimates. For $n = 0, 1, \dots$ let

$$T_n = \sum k^{\varphi} P_k^{-\beta} ,$$

where the sum extends over all k in the range $2^n \leq k < 2^{n+1}$. Thus

$$G(x) << \sum_{2^n \leq x} T_n .$$

Let the inequality (1) hold for all k, where $\gamma > 0$. We successively obtain

$$T_n << \sum 2^{-\frac{\ell}{\gamma}} 2^n 2^{n\varphi} 2^{\ell\beta} = 2^{n(\varphi+1)} \sum 2^{\ell(\beta-\frac{1}{\gamma})} ,$$

where, by Lemma 2, it is sufficient to sum only over these ℓ, with $2^{\ell} << 2^{\gamma n}$. Hence

(7) $$T_n << 2^{n(\varphi+1)} 2^{\{n(\beta\gamma-1)\}} .$$

Summing over all n with $2^n \leq x$, we obtain immediately

(8) $$G(x) << x^f \lg^{\varkappa} x \ ,$$

where $f = max\,(max\,(\beta\gamma, 1) + \varphi, 0)$ and where

$\varkappa = 1$ for $max\,(\beta\gamma, 1) = -\varphi \neq min\,(\beta\gamma, 1)$
and $\varphi > -1 = -\beta\gamma$,

$\varkappa = 2$ for $\beta\gamma = 1 = -\varphi$,

$\varkappa = 0$ otherwise.

These results together with (4) and (6) give full information (up to a certain "logarithmic" gap) about the asymptotic behavior of the function $G(x)$:

Theorem 1. The relation

$$G(x) >> x^{\{\varphi+1\}}$$

always holds. If $\gamma > 0$ and the inequality (1) holds for all k, then

$$G(x) << x^{\{\beta\gamma+\varphi\}}$$

for $\beta\gamma > 1$,

$$G(x) << x^{\{\varphi+1\}} x^{\{\beta\gamma+\varphi\}}$$

for $\beta\gamma \leq 1$. If $\beta\gamma = 1 < -\varphi$, then moreover

$$G(x) << 1 .$$

If $\gamma > 0$ and the inequality (5) holds for infinitely many k, then

$$G(x) = \Omega(x^{\gamma\beta+\varphi})$$

for $\beta\gamma > 1$.

Thus, if $\gamma = \gamma(\alpha_1, \alpha_2, \dots, \alpha_k)$, then

$$\limsup_{x \to +\infty} \frac{\lg G(x)}{\lg x} = \max(\max(\beta\gamma, 1) + \varphi, 0)$$

(for $\gamma = +\infty$ the right hand side is defined by its limit).

Let us note that (8) enables us to prove the convergence of the series

$$\sum_{k=1}^{\infty} k^{\varphi} P_k^{-\beta}$$

for $\max(\beta\gamma, 1) + \varphi < 0$. Relations (4) and (6) give its divergence in the cases $\max(\beta\gamma, -\varphi) \leq 1$ and $\beta\gamma > \max(1, -\varphi)$. If $1 < \beta\gamma = -\varphi$, the series can either converge or diverge depending on the specific value $\alpha_1, \alpha_2, \dots, \alpha_k$. (For example in the case $k = 1$ we can easily construct examples by means of continued fractions.) Here $\gamma = \gamma(\alpha_1, \alpha_2, \dots, \alpha_k)$ and for $\gamma = +\infty$ we interpret all inequalities by limiting processes for $\gamma \longrightarrow +\infty$. Finally, let us note that the "lower exact order" of the function $F(x)$, i.e.,

$$\liminf_{x \to +\infty} \frac{\lg F(x)}{\lg x}$$

is generally unknown (up to certain trivial cases). A similar remark applies for $G(x)$. These questions seem to be more difficult.

§ 3. The sum $H(x)$

Let φ, β, x and A be real numbers, $x > c$,

$A > c$, $\beta \geqq 0$. We consider the sum

$$H(x) = \sum_{h \leqq x} h^{\varrho} \min^{\beta}(A, \frac{1}{P_h}),$$

where we put $\min(A, \frac{1}{B}) = A$ for $B = 0$. Obviously

$$\sum_{h \leqq x} h^{\varrho} << H(x) << A^{\beta} \sum_{h \leqq x} h^{\varrho} ,$$

and hence

(9) $$x^{\varrho+1} << H(x) << A^{\beta} x^{\varrho+1} .$$

Let the numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ be rational and let N denote their least common denominator. Then

(10) $$H(x) = \frac{x^{\varrho+1}}{N} \sum_{j=0}^{N-1} \min^{\beta}(A, \frac{1}{P_j}) + c(\varrho) + O(x^{\varrho})$$

for $\varrho \geqq -1$, where $c(\varrho) = 0$ for $\varrho \geqq 0$ and $c(\varrho)$ is a constant depending only on A, α_j and ϱ, $c(\varrho) << 1$ for $-1 \leqq \varrho < 0$ and

(11) $$H(x) = \sum_{j=0}^{N-1} \min^{\beta}(A, \frac{1}{P_j}) \sum_{h \equiv j (\mathrm{mod}\, N)} h^{\varrho} + O(x^{\varrho+1})$$

for $\varrho < -1$. The proofs are obvious.

Let the inequality (5) hold for infinitely many h, say $h = h_n$, $n = 1, 2, \ldots$ and let $\gamma > 0$. Then

$$H(h_n) \geqq h_n^{\varrho} \min^{\beta}(A, h_n^{\gamma}),$$

hence

(12) $$H(x) = \Omega(x^{\varrho} \min^{\beta}(A, x^{\gamma})) .$$

In the sequel assume that the inequality (1) holds for all $h, \gamma > 0$. We put, as in § 2,

$$T_n = \sum h^{\varphi} \min{}^{\beta}(A, \frac{1}{P_h}) ,$$

where the sum extends over all h in the range $2^n \leq h < 2^{n+1}$. Thus

$$H(x) << \sum_{2^n \leq x} T_n$$

and by Lemmas 1 and 2 we obtain

$$T_n << 2^{n(\varphi+1)} \sum_{2^{\ell} << 2^{\gamma n}} 2^{-\frac{\ell}{\gamma}} \min{}^{\beta}(A, 2^{\ell}) .$$

Now we consider two special cases, according to whether $2^{\gamma n} << A$ or $2^{\gamma n} >> A$. In the first case

$$T_n << 2^{n(\varphi+1)} \sum_{2^{\ell} << 2^{\gamma n}} 2^{\ell(\beta - \frac{1}{\gamma})} ,$$

and hence

(13) $$T_n << 2^{n(\varphi+1)} 2^{\{n(\beta\gamma-1)\}} .$$

In the second case

$$T_n << 2^{n(\varphi+1)} (\sum_{2^{\ell} << A} 2^{\ell(\beta - \frac{1}{\gamma})} + A^{\beta} \sum_{2^{\ell} >> A} 2^{-\frac{\ell}{\gamma}}) ,$$

and hence

(14) $$T_n << 2^{n(\varphi+1)} A^{\{\beta - \frac{1}{\gamma}\}} .$$

From (13) and (14) we obtain

(15) $$H(x) << \sum_{2^n \leq x} 2^{n(\varphi+1)} \min{}^{\{\beta - \frac{1}{\gamma}\}}(A, 2^{\gamma n}) .$$

From (9) - (12) and (15) we obtain:

<u>Theorem 2</u>. The relations

$$x^{\{\varphi+1\}} << H(x) << A^{\beta} x^{\{\varphi+1\}}$$

always hold. If the numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ are rational and N is their least common denominator, then we have the relations (10) and (11). If $\gamma > 0$ and the inequality (1) holds for all k, then

$$H(x) << \min^{\{\beta\gamma+\varphi\}}(x, A^{\frac{1}{\gamma}}) \max^{\{\varphi+1\}}(2, xA^{-\frac{1}{\gamma}})$$

for $\beta\gamma > 1$,

$$H(x) << x^{\{\varphi+1\}} \min^{\{\beta\gamma-1\}}(x, A^{\frac{1}{\gamma}})$$

for $\beta\gamma \leqq 1$. If $\beta\gamma = 1 < -\varphi$ then moreover $H(x) << 1$. Finally, if the inequality (5) holds for infinitely many k, then

$$H(x) = \Omega(x^{\varphi} \min^{\beta}(A, x^{\gamma})) .$$

The "exact order" of the function $H(x)$ generally depends on the relation between x and A. If $\beta\gamma \leqq 1$ we have however

$$\limsup_{x \to +\infty} \frac{\lg H(x)}{\lg x} = \max(\varphi + 1, 0)$$

and the same relation holds in the case $\lg A = o(\lg x)$. The relation (12) can easily be improved if $A = A(x)$ is an increasing continuous function, the inequality (5) with $\gamma > 0$ holds for infinitely many k, say $k = k_m$, $m = 1, 2, \ldots$, and $A(x) \leqq x^{\gamma}$. Then for $x_m = A^{-1}(k_m^{\gamma})$

we get

$$H(x_n) \geq k_n^{\varphi} \min^{\beta}(A(x_n), k_n^{\gamma}) = k_n^{\varphi+\beta\gamma}$$

and hence $H(x) = \Omega(A^{\beta+\frac{\varphi}{\gamma}}(x))$. In this case, for $\beta\gamma > -\varphi \geq 1$, our theorem yields

$$H(x) = O(A^{\beta+\frac{\varphi}{\gamma}}(x)),$$

provided that the inequality (1) holds for all k, etc.

In the important case, when A is independent on x, we have the following corollary.

Corollary. Let $\varphi + 1 < 0$ and let, for a certain $\gamma > 0$, the inequality (1) hold for all k. Then

$$H_A = \sum_{k=1}^{\infty} k^{\varphi} \min^{\beta}(A, \frac{1}{P_k}) \asymp 1$$

for $\beta\gamma + \varphi < 0$,

$$1 << H_A << lg\, A$$

for $\beta\gamma + \varphi = 0$ and

$$1 << H_A << A^{\beta+\frac{\varphi}{\gamma}}$$

for $\beta\gamma + \varphi > 0$. If the inequality (5) holds for infinitely many k (say $k = k_n$), $\gamma > 0$, then there is a sequence of the numbers $A = A_n$ (namely $A_n = k_n^{\gamma}$) such that

$$H_{A_n} >> A_n^{\beta+\frac{\varphi}{\gamma}} .$$

Let $\varphi = -1$ and let, for a certain $\gamma > 0$, the inequality (1) hold for all k. Then

$$lg\, x << H(x) << A^{(\beta-\frac{1}{\gamma})} lg\, x$$

for $\beta\gamma \leq 1$ and

$$\lg x << H(x) << \Lambda^{(\beta - \frac{1}{\gamma})} \lg \frac{x}{\Lambda^{\frac{1}{\gamma}}}$$

for $\beta\gamma > 1$, provided $x^{\gamma} >> \Lambda$.

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