Werk

Label: Article

Jahr: 1972

PURL: https://resolver.sub.uni-goettingen.de/purl?316342866_0013|log68

Kontakt/Contact

Digizeitschriften e.V.
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

info@digizeitschriften.de

Commentationes Mathematicae Universitatis Carolinae

13,4 (1972)

ON GENERAL CONCEPT OF BASIC SUBGROUPS

Pavel JAMBOR, Jindřich BEČVAŘ, Praha

0. Introduction. The notion of basic subgroups has been introduced by Kulikov [5] and developed by Fuchs [3]. The purpose of this paper is to build up the general concept of basic subgroups of abelian groups which can be extended to any Abelian category. In the section 1, there are presented some fundamental properties of the basic subgroups, in particular the proposition 1.7 shows the equivalence of our definition with the Kulikov's one in reduced torsion abelian groups and the corollary 1.12 describes the structure of alg. compact groups with respect to its basic subgroups. The section 2 develops the concept of basic decompositions with respect to the basic subgroups, i.e. there is discussed the question of an embedding of a given ab. group into a direct product of its basic components.

Throughout the paper a group G always stands for an abelian group. Concerning the terminology and notation, we refer to [3], 282. Otherwise, if G is a group then G_t, G_p, G_D and G^1 are the torsion part, p-component of G_t, divisible part and the first Ulm's subgroup of G respec-

---

AMS, Primary: 20K25 Ref. Ž. 2.722.1

tively. We shall frequently use the following notation:

$\mathbb{N}$ - natural numbers,

$\mathbb{N}^+ = \{ n \in \mathbb{N} ; n > 0 \}$,

$\mathbb{P}$ - the set of all prime numbers,

$\hat{G}$ - the Z-adic completion of G ,

(n, m) - the greatest common divisor of n and m .

If $\{ G_\alpha ; \alpha \in \Lambda \}$ is a family of groups then $\coprod_{\alpha \in \Lambda} G_\alpha$ stands for the direct sum and any subgroup H , where $\coprod G_\alpha \subset H \subset \prod G_\alpha$ is called the interdirect sum of $\prod_{\alpha \in \Lambda} G_\alpha$. By a superdecomposable group we mean a group with no nonzero indecomposable direct summand (see [1]). For a $K \subset \mathbb{P}$, we define that H is a K-pure subgroup of G if $p^n H = H \cap p^n G$, for $\forall p \in K$, $\forall n \in \mathbb{N}$ and G is K-divisible if $G = pG$, for $\forall p \in K$.

1. An outline of the theory.

Definition 1.1. We shall say that B is a basic subgroup of a group G if

(i) $B = \langle \{ G_\alpha ; \alpha \in \Lambda \} \rangle$, where $0 \neq G_\alpha$ is an indecomposable subgroup of G , for $\forall \alpha \in \Lambda$.

(ii) $\langle \{ G_\alpha ; \alpha \in K \} \rangle = \coprod_{\alpha \in K} G_\alpha$ and $\coprod_{\alpha \in K} G_\alpha$ is a direct summand of G for each finite $K \subset \Lambda$.

(iii) The family $\{ G_\alpha ; \alpha \in \Lambda \}$ is maximal with respect to the conditions (i) and (ii).

The family $\{ G_\alpha ; \alpha \in \Lambda \}$ is called the basic system of G corresponding to B .

Remark 1.2. The condition 1.1 (ii) implies that B is a pure subgroup of G and $B = \coprod_{\alpha \in \Lambda} G_\alpha$. In the following we shall write $B_F = \coprod_{\alpha \in \Lambda_F} G_\alpha$ and $B_R = \coprod_{\alpha \in \Lambda_R} G_\alpha$, where $\Lambda_F = \{\alpha \in \Lambda ;\ G_\alpha$ is torsion-free$\}$ and $\Lambda_R = \{\alpha \in \Lambda ;\ G_\alpha$ is reduced$\}$. Similarly we shall use the symbols Λ_t, Λ_{π} and Λ_D for B_t, B_{π} and B_D respectively.

Proposition 1.3. Every group contains a basic subgroup.

Proof. Let us denote the set of all the families $\{G_\alpha ; \alpha \in \Lambda\}$ satisfying 1.1 (i) and (ii) in a group G by $D(G)$. Obviously $D(G) \neq \emptyset$. Let $\{D_\gamma ; \gamma \in \Gamma\} \subset D(G)$ be a chain with respect to the inclusion and $C = \bigcup_{\gamma \in \Gamma} D_\gamma$. If $G_{\alpha_1}, \dots, G_{\alpha_m} \in C$ then there is a $\gamma \in \Gamma$ such that $G_{\alpha_1}, \dots, G_{\alpha_m} \in D_\gamma$. Hence $C \in D(G)$ and the Zorn's lemma immediately implies the existence of B. q.e.d.

Proposition 1.4. Let B be a basic subgroup of G. Then

(i) If $G = H \oplus W$, $B \subset W$, then H is a superdecomposable group.

(ii) G/B contains no nonzero cylic direct summand.

(iii) B_t is a basic subgroup of G_t.

(iv) G/B_t is a splitting group, where $(G/B_t)_t = G_t/B_t$ is a divisible group.

(v) G/B is a splitting group, where $(G/B)_t = (G_t + B)/B$ is a divisible group.

Proof. Let $\{G_\alpha ; \alpha \in \Lambda\}$ be a basic system of G corresponding to B.

(i) If H' is an indecomposable direct summand of H then the family $\{H'\} \cup \{G_\alpha ; \alpha \in \Lambda\}$ obviously satisfies 1.1 (ii) and the maximality of $\{G_\alpha ; \alpha \in \Lambda\}$ implies $H' = 0$.

(ii) If $G/B = (G_1/B) \oplus (G_2/B)$, where G_1/B is a cylic group then $G_1 = B \oplus G_1'$ since B is a pure subgroup of G_1. Hence $G = G_1' \oplus G_2$, where $B \subset G_2$ and (i) yields $G_1' = 0$.

(iii) Obviously $\{G_\alpha ; \alpha \in \Lambda_t\}$ satisfies 1.1 (i) and (ii) in G_t. If $\{H\} \cup \{G_\alpha ; \alpha \in \Lambda_t\}$ satisfies the conditions 1.1 (i) and (ii) in G_t then $\{H\} \cup \{G_\alpha ; \alpha \in \Lambda\}$ satisfies 1.1 (i) and (ii) in G. For, if $K_t \subset \Lambda_t$ and $K_F \subset \Lambda_F$ are finite then $G = \coprod_{\alpha \in K_F} G_\alpha \oplus W$ and obviously $G_t \subset W$. Since $\coprod_{\alpha \in K_t} G_\alpha \oplus H$ is a direct sum of a divisible and a bounded group and it is a pure subgroup of W, we can write $G = \coprod_{\alpha \in K_F} G_\alpha \oplus \coprod_{\alpha \in K_t} G_\alpha \oplus H \oplus W'$. Hence it contradicts the maximality of the family $\{G_\alpha ; \alpha \in \Lambda\}$.

(iv) By (iii) and (ii) G_t / B_t is divisible and consequently $G/B_t = (G_t/B_t) \oplus (G'/B_t)$, where $(G'/B_t) \cong (G/G_t)$ is torsion-free.

(v) $(G_t + B)/B \cong G_t/(B \cap G_t) = G_t/B_t$ and by (iv) $G/B = ((G_t + B)/B) \oplus (G'/B)$. If $g' \in G'$ and $mg' \in B$ then $mg' = mb$ for some $b \in B$ since B is pure in G'. Hence $(g' - b) \in G_t$ i.e. $g' \in (G_t + B) \cap G' = B$ and consequently G'/B is torsion-free. q.e.d.

<u>Proposition 1.5</u>. $B = B_D \oplus B_R$ is a basic subgroup of a group G iff $B_D = G_D$ and B_R is a basic subgroup

of some direct complement of G_D .

Proof. (i) Let B be a basic subgroup of a group G and $\{G_\alpha ; \alpha \in \Lambda\}$ be a corresponding basic system of G . Then $B_R \cap G_D = 0$. For, if $g \in B_R \cap G_D$ then $g \in \in \bigcap_{n \in N^+} nB_R = \coprod_{\alpha \in \Lambda_R} (\bigcap_{n \in N^+} nG_\alpha)$ since B_R is pure in G . On the other hand G_α is either reduced torsion-free or torsion cyclic for $\forall \alpha \in \Lambda_R$. Hence $\bigcap_{n \in N^+} nG_\alpha = 0$, for $\forall \alpha \in \Lambda_R$. Since G_D is an absolute direct summand there is a direct complement R of G_D containing B_R and by 1.4 (i), $B_D = G_D$. Now, it is sufficient to show that $\{G_\alpha ; \alpha \in \Lambda_R\}$ is a basic system of R . For, the conditions 1.1 (i),(ii) are obvious and 1.1 (iii) immediately follows from the maximality of $\{G_\alpha ; \alpha \in \Lambda\}$ in G and from the decomposition $G = B_D \oplus R$.

(ii) Conversely, let $G = G_D \oplus R$, B_R be a basic subgroup of R and $\{G_\alpha ; \alpha \in \Lambda_R\}$ be the corresponding basic system of R . Since G_D is a direct sum of nonzero indecomposable subgroups, say $G_D = \coprod_{\alpha \in \Lambda_D} G_\alpha$ then the family $\{G_\alpha ; \alpha \in \Lambda_D \cup \Lambda_R\}$ obviously satisfies the conditions 1.1 (i),(ii) in G . Let $\{G_\alpha ; \alpha \in \Lambda_D \cup \Lambda_R\} \cup \{H\}$ be a family of subgroups satisfying 1.1 (i),(ii) in G then H is obviously a reduced subgroup and since G_D is an absolute direct summand of G there is a direct complement R' of G_D , containing $B_R \oplus H$. If $r' \in R'$ then $r' = (d + r) \in G_D \oplus R$ and the map $\varphi : \begin{matrix} R' \longrightarrow R \\ r' \longmapsto r \end{matrix}$ is an isomorphism R' onto R which is the identity on B_R . Hence $\varphi(B_R \oplus H) = B_R \oplus \varphi(H)$ and $\{G_\alpha ; \alpha \in \Lambda_R\} \cup \{\varphi(H)\}$

satisfies the conditions 1.1 (i),(ii) in $\mathfrak{R}$, which contradicts the maximality of $\{G_\alpha ; \alpha \in \Lambda_R\}$ in $\mathfrak{R}$. q.e.d.

Proposition 1.6. Let G be either cotorsion or a group, where the only indecomposable direct summands are either divisible or torsion cyclic and $B \subset G$ be a subgroup satisfying 1.1 (i),(ii) in G . Then B is basic in G iff $G = H \oplus W$, $B \subset W$ implies that H is superdecomposable.

Proof. With respect to 1.4 (i) we shall prove only the sufficient condition. Let $\{G_\alpha ; \alpha \in \Lambda\}$ be a family corresponding to B . If there is a larger family $\{H\} \cup \cup \{G_\alpha ; \alpha \in \Lambda\}$ satisfying 1.1 (i),(ii) in G then by the hypothesis H is alg. compact. By 1.2 $H \oplus B$ is pure in G and consequently $(H \oplus B)/B$ is pure in G/B . Hence $G/B = ((H \oplus B)/B) \oplus (G'/B)$ and consequently $G = H \oplus G'$, where $B \subset G'$. So, H being superdecomposable yields a contradiction , q.e.d.

Proposition 1.7. Let B be a subgroup of a reduced torsion group G . Then B is basic in G iff

(i) B is a direct sum of cyclic subgroups of prime power orders,

(ii) B is a pure subgroup,

(iii) G/B is divisible.

Proof. If B is a basic subgroup then by 1.1 (i),1.2 and 1.4 (iv) the conditions (i),(ii) and (iii) are satisfied. Conversely, (i) implies 1.1 (i) and since every finite direct sum of groups of a prime power orders is bounded and by (ii) is pure in G, 1.1 (ii) is satisfied. If $G = H \oplus W$, $B \subset W$ then $G/B \cong H \oplus (W/B)$ and (iii) implies that $H = 0$. Hence by 1.6, B is a basic subgroup of G. q.e.d.

Proposition 1.8. A subgroup $B = \coprod_{p \in P} B_p$ of a torsion group G is basic in G iff B_p is a basic subgroup of G_p, for $\forall p \in P$.

Proof. If B is a basic subgroup of G then B_p obviously satisfies 1.1 (i),(ii) in G_p. Let $G_p = H \oplus W$, $B_p \subset W$; then $G = H \oplus W \oplus \coprod_{q \in P, q \neq p} G_q$, $B \subset W \oplus \coprod_{q \in P, q \neq p} G_q$ and 1.6 implies that $H = 0$ and B_p is a basic subgroup of G_p. Conversely, B obviously satisfies 1.1 (i) and since $G = \coprod_{p \in P} G_p$ and each B_p satisfies 1.1 (ii) in G_p, B satisfies 1.1 (ii) in G, too. If $G = H \oplus W$, $B \subset W$ then $G_p = H_p \oplus W_p$, where $B_p \subset W_p$ and 1.6 implies that $H_p = 0$, for $\forall p \in P$, i.e. $H = 0$ and B is a basic subgroup of G. q.e.d.

Proposition 1.9. If G is either alg. compact or an adjusted group and B a basic subgroup of G, then G/B is divisible.

Proof. (i) Let G be alg. compact. By 1.5, $B_D = G_D$ and there is a subgroup $G' \subset G$ such that $G = G' \oplus G_D$ and B_R is a basic subgroup of G'. Obviously $G/B \cong$ $\cong G'/B_R$ and G' is the reduced alg. compact group. Hence by [3], 39.1, 163, G' is complete in the Z-adic topology and since B_R is pure in G', $G' = H \oplus (\bigcap_{n \in N^+} (B_R + nG'))$ by [3], ex.2, 166. On the other hand 1.4 (i) and [3], 40.4, 169 imply $H = 0$. Hence $G' = B_R + nG'$, for $\forall n \in N^+$.

(ii) Let G be an adjusted group. By [3], 55.1,237, G/G_t is divisible. Since $B_t = B$, 1.4 (iv) implies that G_t/B is divisible. Hence $G/B = H \oplus (G_t/B)$, where $H \cong G/G_t$. q.e.d.

Corollary 1.10. If G is a cotorsion group then there exists a basic subgroup $B \subset G$ such that G/B is divisible.

Proof. By [3], 55.5, 238, $G = A \oplus C \oplus G_D$, where C is the adjusted part and A is alg. compact. If B_A and B_C are basic subgroups of A and C respectively then $B_A \oplus B_C \oplus G_D$ satisfies 1.1 (i),(ii) in G and there is a basic subgroup B of G such that $B_A \oplus B_C \oplus G_D \subset B$. By 1.9, $G/(B_A \oplus B_C \oplus G_D) \cong (A/B_A) \oplus (C/B_C)$ is divisible hence G/B is divisible, too. q.e.d.

Theorem 1.11. Let B be a basic subgroup of a group G. If $G^1 = 0$ and G/B is divisible then G can be embedded as a pure subgroup in $\hat{B}$. Moreover, if G is torsion-free then $\hat{B} \cong \mathrm{Ext}(Q/Z, B)$.

Proof. Consider the following map $\varphi: G \longrightarrow \hat{B}$. Since G/B is divisible, there is a sequence $\{b_n\}_{n=1}^{\infty}$ for $\forall g \in G$, such that $(g - b_n) \in n!\,G$ and $b_n \in B$, for $\forall n \in N^+$, therefore $g = \lim \{b_n\}_{n=1}^{\infty}$ in Z-adic topology of G. Now, let $\varphi(g) = \lim \{b_n\}_{n=1}^{\infty} \in \hat{B}$. Since G is Hausdorff in Z-adic topology, $\hat{B}$ is Hausdorff by [4], 29, 27 and B is pure in G, φ is a well-defined homomorphism. If $\lim \{b_n\}_{n=1}^{\infty} = g \in G$ and $\lim \{b_n\}_{n=1}^{\infty} = 0 \in \hat{B}$ then $b_n \in n\hat{B}$ for $\forall n \in N^+$

$\forall k \geq n$. By [4], 30, 28, B is pure in $\hat{B}$ hence $b_k \in$ $\in nB \subset nG$ and consequently $g = 0$. Therefore, φ is an embedding and without loss of generality we can assume that $G \subset \hat{B}$. Let $g = n\hat{b}$, where $g \in G$ and $\hat{b} \in \hat{B}$. The divisibility of G/B implies that $g = b + nh$, for some $b \in B$, $h \in G$. Therefore $b = n(\hat{b} - h) = nb'$, for some $b' \in$ $\in B$, since B is pure in $\hat{B}$. Hence $g = n(b' + h)$. Moreover, if G is torsion-free then $\hat{B}$ is torsion-free, complete in the Z-adic topology ([4], 29, 27) and reduced. Hence by [3], 39.1, 163 $\hat{B}$ is cotorsion and $\hat{B}/B$ is torsion-free, divisible by [4], 30, 28. The following exact sequence

$$0 = \mathrm{Hom}(Q/Z, \hat{B}/B) \to \mathrm{Ext}(Q/Z, B) \to \mathrm{Ext}(Q/Z, \hat{B}) \cong \hat{B} \to \mathrm{Ext}(Q/Z, \hat{B}/B) = 0$$

immediately implies the desired result. q.e.d.

Corollary 1.12. Let B be a basic subgroup of a group G, where $G^1 = 0$. Then G is alg. compact iff $G \cong \hat{B}$.

Proof. Let G be alg. compact. According to 1.9, G/B is divisible and hence it is sufficient to show that the embedding $\varphi : G \longrightarrow \hat{B}$ from 1.11 is onto. If $\hat{b} \in \hat{B}$ then $\hat{b} = \lim \{b_n\}_{n=1}^{\infty}$ in $\hat{B}$ for some $\{b_n ; n \in \mathbb{N}^+\} \subset B$. Since $\{b_n\}_{n=1}^{\infty}$ is a Cauchy-sequence in B i.e. in G and by [3], 39.1, 163 G is complete in the Z-adic topology i.e. there is a $g \in G$ such that $\varphi(g) = \hat{b}$. The converse of the corollary immediately follows from [3], 39.1, 163. q.e.d.

Proposition 1.13. There are neither nonzero cotorsion, slender nor nonzero cotorsion, $\aleph_1$-free groups.

Proof. If $G \neq 0$ is a cotorsion, slender group then by [4], 154, 111 G is torsion-free, reduced and there is no copy of p-adic integers in G. Hence by [3], 54.5, 235 G is alg. compact and([3], 40.4, 169) implies a contradiction.

If $G \neq 0$ is a cotorsion, $\aleph_1$-free group then G is torsion-free, reduced and consequently G is alg. compact. Hence by [3], 40.4, 169 G contains a copy of p-adic integers J_p and since $Q_p \subset J_p$, it contradicts the hypothesis. q.e.d.

2. On basic decompositions.

Definition 2.1. Let B be a subgroup of G satisfying 1.1 (i),(ii). Then the intersection of all the direct complements of $\coprod_{\alpha \in K} G_\alpha$, where K runs through the finite subsets of Λ, is called the B-residual subgroup of G, denoted by R_B^G.

Proposition 2.2. Let B be a subgroup of G satisfying 1.1 (i),(ii). Then R_B^G is the greatest fully invariant subgroup disjoint with B.

Proof. Denote by W_K the intersection of all the direct complements of $\coprod_{\alpha \in K} G_\alpha$, for a finite $K \in \Lambda$. By [3], 9.6, 48, W_K is fully invariant subgroup and R_B^G as an intersection of fully invariant subgroups is fully invariant, too, and obviously $R_B^G \cap B = 0$. If A is a fully invariant subgroup such that $A \cap B = 0$ and $G = (\coprod_{\alpha \in K} G_\alpha) \oplus W$ a direct decomposition for a finite $K \subset \Lambda$, then by [3], 9.3, 47 $A \subset W$. Hence $A \subset R_B^G$. q.e.d.

The proof of the following proposition is straightforward and hence omitted.

Proposition 2.3. Let G be a group. Then the following are equivalent:

(i) 0 is a basic subgroup,

(ii) G is superdecomposable,

(iii) 0 is the unique basic subgroup,

(iv) if B is a basic subgroup, then $R_B^G = G$,

(v) there is a basic subgroup B such that $R_B^G = G$.

Lemma 2.4. Let D and M be subgroups of a group G such that M is D-high in G. If $y \in G \setminus D \oplus M$ and $ky \in M$, for some $k \in \mathbb{N}^+$ then there exists a $m \in \mathbb{N}^+$ such that $1 < m < k$, $m \mid k$ and $my \in D \oplus M \setminus M$. In particular, k is not a prime.

Proof. Since $D \cap \langle M, y \rangle \neq 0$, consider a $d = m + ly$, where $d \in D$, $d \neq 0$, $m \in M$ and $l \in \mathbb{N}^+$. Let $m = (l, k)$, $k = k'm$ and $l = l'm$. Since $ul' + vk' = 1$, for some $u, v \in \mathbb{Z}$ we get $ud = um + my - vky$ and hence $my \in M \oplus D$ and consequently $m > 1$. If $my \in M$ then obviously $ud = 0$, i.e. $\sigma(d) \mid u$. Furthermore, $k'd = k'm + l'ky$, so $k'd = 0$ and $\sigma(d) \mid k'$. Hence $\sigma(d) = 1$ implies a contradiction and $my \notin M$, $m < k$ follows immediately. q.e.d.

Next we shall use the following notation. If $B = \coprod_{\alpha \in \Lambda} G_\alpha$, where each G_α is a nonzero reduced indecomposable torsion group then $B_{p,m}$ stands for $\coprod_{\alpha \in \Lambda_{p,m}} G_\alpha$, where $\Lambda_{p,m} = \{\alpha \in \Lambda ;$ $G_\alpha \cong Z(p^m)\}$. Consider $K_B = \{p \in P ; B_p \neq 0\}$ in the natural order. In case that $\mathbb{K}_B = \{p_1, \dots, p_\ell\}$, take $p_s = 1$ for each $s > \ell$. Define the following sequence from $\mathbb{N}^+$ by $k_0 = 1$ and $k_{n+1} = k_n \cdot p_1 \cdots p_{n+1}$ and the corresponding sequence of subgroups by $B_0 = 0$, $B_{n+1} = \coprod_{i=1}^{n+1} B_{p_i, n+2-i}$, for each $n \in \mathbb{N}$. Then obviously $B = \coprod_{n \in \mathbb{N}} B_n$ and $k_n B \subset \coprod_{i > n} B_i$.

Theorem 2.5. Let G be a group, $B \subset G$ a reduced torsion subgroup satisfying 1.1 (i),(ii) in G and $F \subset G$ be a $\mathbb{K}$-pure torsion-free subgroup, where $\mathbb{K}_B \subset \mathbb{K} \subset \mathbb{P}$. Then

1) For $\forall n \in \mathbb{N}$, there is a subgroup $W_n \subset G$ such that

(i) $G = B_1 \oplus \dots \oplus B_n \oplus W_n$,

(ii) $W_n = B_{n+1} \oplus W_{n+1}$,

(iii) W_{n+1} is a B_{n+1}-high subgroup of W_n containing $(\coprod_{i > n+1} B_i + F + k_{n+1} G)$.

2) There is a homomorphism $\varphi : G \longrightarrow \prod_{\substack{p \in \mathbb{K}_B \\ n \in \mathbb{N}^+}} B_{p,n}$ whose image is an interdirect sum of $\prod B_{p,n}$ and

(i) $G^1 \subset \bigcap_{\substack{p \in \mathbb{K}_B \\ n \in \mathbb{N}}} p^n G = \bigcap_{n \in \mathbb{N}} k_n G \subset R_B^G \subset \ker \varphi = \bigcap_{n \in \mathbb{N}} W_n$,

(ii) $\bigcap_{n \in \mathbb{N}} (F + k_n G) \subset \bigcap_{\substack{p \in \mathbb{K}_B \\ n \in \mathbb{N}}} (F + p^n G) \subset \ker \varphi$.

3) If $G/_{B \oplus F}$ is $\mathbb{K}$-divisible then

$\ker \varphi = \bigcap_{\substack{p \in \mathbb{K} \\ n \in \mathbb{N}}} (F + p^n G) = \bigcap_{n \in \mathbb{N}} (F + k_n G)$ and

$\varphi(G)$ is $\mathbb{K}$-pure in $\prod B_{p,n}$.

In particular, if $G/_B$ is $\mathbb{K}$-divisible then

$R_B^G = \ker \varphi = \bigcap_{\substack{p \in \mathbb{K} \\ n \in \mathbb{N}}} p^n G$.

Proof. 1) $B_n \cap (\coprod_{i>n} B_i + F + k_n G) = 0$, for $\forall n \in \mathbb{N}$. For, let $b = b' + f + k_n g$, where $b \in B_n$, $b' \in \coprod_{i>n} B_i$, $f \in F$, $g \in G$ and $v(b - b') = m$, then $m = \prod p_i^{\alpha_i}$, where each $p_i \in \mathbb{K}_B$. Since F is $\mathbb{K}_B$-pure there is a $f' \in F$ such that $-mf = mk_n g = mk_n f'$. Hence $f = -k_n f'$, $b - b' = k_n(g - f') = k_n b'' \in \coprod_{i>m} B_i$ since B is pure in G and consequently $b = 0$.

For $n = 0$, define $W_0 = G$ and W_1 as a B_1 -high subgroup of W_0 containing $(\coprod_{i>1} B_i + F + k_1 G)$. Obviously $W_0 \supset W_1 \oplus B_1$ and since $p_1 g \in W_1$, for every $g \in W_0$, 2.4 implies $W_0 = B_1 \oplus W_1$. Suppose that the theorem holds for an $n - 1 \geq 0$. Hence $W_n \supset B_{n+1} \oplus (\coprod_{i>n+1} B_i + F + k_{n+1} G)$. Denote by W_{n+1} the B_{n+1} -high subgroup of W_n containing $(\coprod_{i>n+1} B_i + F + k_{n+1} G)$. Let $g \in W_n \setminus B_{n+1} \oplus W_{n+1}$, then obviously $k_{n+1} g \in W_{n+1}$.
By 2.4, let r_1 be the greatest natural number such that $1 < r_1 < k_{n+1}$, $r_1 \mid k_{n+1}$ and $r_1 g \in B_{n+1} \oplus W_{n+1} \setminus W_{n+1}$.

Then $r_1 g = b + w$, where $b \in B_{n+1}$, $w \in W_{n+1}$ and there is an $i \in \{1, \dots, n+1\}$ such that $p_i r_1 g \in W_{n+1}$. Hence $p_i r_1 g - p_i w = p_i b \in W_{n+1}$ and consequently $p_i b = 0$, i.e. $b \in B_{p_i, n+2-i}$ and $b = p_i^{n+1-i} b'$, for some $b' \in B_{p_i, n+2-i}$. Let $r_1 = p_i^a r_1'$, where $(r_1', p_i) = 1$, then $a \leq n + 1 - i$. Obviously $b' = r_1' b''$, where $b'' \in B_{p_i, n+2-i}$. Hence $b = p_i^{n+1-i} r_1' b'' = r_1 b_1$,

$g - b_1 \notin B_{n+1} \oplus W_{n+1}$ and $r_1(g - b_1) = w \in W_{n+1}$. By the repeated use of 2.4 we get finite sequences $b_1, \dots, b_s$ and $k_{n+1} > r_1 > \dots > r_s > 1$, where $r_s \in \mathbb{P}$, such that $(g - b_1 - \dots - b_s) \notin B_{n+1} \oplus W_{n+1}$ and $r_s(g - b_1 - \dots - b_s) \in W_{n+1}$, which contradicts 2.4.

2) Consider the following map

$$\varphi : G \longrightarrow \prod B_{p,n}$$

$$g \longmapsto (b_{p_1,1}; b_{p_1,2}, b_{p_2,1}; b_{p_1,3}, b_{p_2,2}, b_{p_3,1}; \dots),$$

where $b_{p_i,j}$ are the direct components of g in the decompositions $G = B_{p_1,1} \oplus \dots \oplus (\coprod_{i=1}^{n} B_{p_i,n+1-i}) \oplus W_n$, for each $n \in \mathbb{N}$. With respect to the equality $W_n = B_{n+1} \oplus W_{n+1}$, φ is a well-defined map and a homomorphism as well.

Since $B = \coprod_{\substack{p \in K_B \\ n \in \mathbb{N}^+}} B_{p,n}$ and φ/B is an identity, $im\, \varphi$ is an interdirect sum of $\prod B_{p,n}$. For the rest it is sufficient to prove 2) (ii). In particular, $\bigcap_{\substack{p \in K_B \\ n \in \mathbb{N}}} (F + p^n G) \subset \ker \varphi$. For, if $g \in \bigcap (F + p^n G)$ and $n \in \mathbb{N}^+$ then $g = f_i + p_i^{n+1-i} h_i$, for some $f_i \in F$, $h_i \in G$ and an arbitrary $i = 1, \dots, n$ ($k_n = \prod_{i=1}^{n} p_i^{n+1-i}$, $p_i \in K_B$). Since $G = B_1 \oplus \dots \oplus B_n \oplus W_n$, we can write $h_i = b_i + w_i$, where $b_i \in \coprod_{j=1}^{n} B_j$ and $w_i \in W_n$, $i = 1, \dots, n$. Now, $g = p_i^{n+1-i} b_i + (f_i + p^{n+1-i} w_i)$, $i = 1, \dots, n$ and since $b = p_i^{n+1-i} b_i = p_j^{n+1-j} b_j$, for $i, j = 1, \dots, n$, $b = 0$ and $g \in W_n$.

3) $\ker\varphi \subset \bigcap_{\substack{j,n\in \mathbb{N}\\ p\in \mathbb{K}}} (F + p^j k_n G)$. For, if $g \in \ker\varphi$ then for $\forall(p\in \mathbb{K})\forall(j,n\in \mathbb{N})$, $g = b + f + p^j k_n h$ for some $b\in B$, $f\in F$ and $h\in G$ since $G/B\oplus F$ is $\mathbb{K}$-divisible. Since $g\in W_n$, $b \in \coprod_{i=n+1}^{n+t} B_i$, for some $t\in \mathbb{N}$ and similarly $g = b' + f' + p^j k_{n+t} h'$, where $b' \in \coprod_{i\ge n+t} B_i$, $f'\in F$ and $h'\in G$. If $o(b-b') = m$ then $m = \prod p_i^{\alpha_i}$, where each $p_i \in \mathbb{K}_B$ and since F is $\mathbb{K}$-pure there is a $f''\in F$ such that $m(f-f') = mp^jk_n(h''-h) = mp^jk_n f''$, where $k_n h'' = k_{n+t} h'$. Hence $f - f' = p^j k_n f''$, $b - b' = p^j k_n(h''-h-f'') = p^jk_n b''$, for some $b''\in B$, since B is pure in G and consequently $b = p^j k_n b_1$, for some $b_1 \in \coprod_{i\ge n}^{n+t} B_i$, implies $g\in F + p^j k_n G$. Hence $\ker\varphi \subset \bigcap_{\substack{n\in \mathbb{N}\\ p\in \mathbb{K}}}(F+p^nG)$, $\ker\varphi\subset\bigcap_{n\in \mathbb{N}}(F+k_nG)$.

$\varphi(G)/\varphi(B) \cong G/\ker\varphi\oplus B$ is $\mathbb{K}$-divisible, i.e. $\varphi(G) = \varphi(B) + p^n\varphi(G)$, for $\forall(p\in \mathbb{K}),\forall(n\in \mathbb{N})$. Let $p^n x = h \in \varphi(G)$, for some $x\in \prod B_{p,n}$ and $p\in \mathbb{K}$, then $h = b + p^n h'$, for some $b\in\varphi(B)$ and $h'\in\varphi(G)$. Since $\varphi(B) = \coprod B_{p,n}$ is pure in $\prod B_{p,n}$, there is a $b'\in\varphi(B)$ such that $b = p^n b'$ and hence $h = p^n(b'+h')$.
q.e.d.

Definition 2.6. Under the conditions of the theorem 2.5 we shall call the sequence $\{W_n;\ n\in \mathbb{N}\}$ the B-sequence of G and the corresponding homomorphism φ the

B -homomorphism of G .

Corollary 2.7. If G is either a reduced torsion or an adjusted group and B is a basic subgroup of G , then G/G^1 is isomorphic to a pure interdirect sum H of $\prod_{\substack{p \in K_B \\ n \in \mathbb{N}^+}} B_{p,n}$ .Moreover, in case of an adjusted group, H is a direct summand.

Proof. With respect to 1.7, 1.9 and 2.5 it is sufficient to show that H is a direct summand provided that G is adjusted. By [3], 54.3, 235 $H \cong G/G^1$ is alg. compact and hence a direct summand . q.e.d.

Proposition 2.8. Let B be a basic subgroup of a reduced group G . Then $(R_B^G)_t = (R_{B_t}^G)_t = (\ker \varphi)_t = G_t^1$ for every B_t -homomorphism φ of G .

Proof. Let $\{W_n ; n \in \mathbb{N}\}$ be a B_t -sequence of G . Since G_t is fully invariant in G, $G_t = B_1 \oplus \ldots \oplus B_n \oplus (W_n \cap G_t)$, for $\forall n \in \mathbb{N}$ and the divisibility of G_t / B_t (see 1.4 (iv)) implies $W_n \cap G_t = \coprod_{i > n} B_i + p^n G_t$. Hence $(\ker \varphi)_t = G_t \cap (\bigcap_{n \in \mathbb{N}} W_n) = \bigcap (W_n \cap G_t) = \ker \varphi'$,where $\varphi' = \varphi / G_t$ and 2.5 and 1.4 (iv) imply $\ker \varphi' = G_t^1$.
The converse inclusion $G_t^1 \subset (R_B^G)_t \subset (R_{B_t}^G)_t \subset (\ker \varphi)_t$ follows immediately from 2.2 and 2.5 . q.e.d.

Proposition 2.9. The adjusted part of $\prod_{\substack{p \in P \\ n \in \mathbb{N}}} H_{p,n}$,where $H_{p,n} = \coprod_{\alpha \in A_{p,n}} H_\alpha$, $H_\alpha \cong Z(p^n)$, for $\forall \alpha \in A_{p,n}$,is a minimal direct summand of $\prod H_{p,n}$ containing $\coprod H_{p,n}$.

Proof. By 1.8, 1.7 and [2], 29.6, 100 , $\coprod H_{p,n}$ is a basic subgroup of $(\prod H_{p,n})_t$. Let G be the adjusted part of $\prod H_{p,n}$, then $\coprod H_{p,n}$ is a basic subgroup of G . For, $\coprod H_{p,n} \subset (\prod H_{p,n})_t \subset G$ and $\coprod H_{p,n}$ obviously satisfies 1.1 (i),(ii) in G . If $A \oplus \coprod H_{p,n}$ also satisfies 1.1 (i),(ii) in G , where A is a nonzero indecomposable subgroup of G , then $A \oplus \coprod H_{p,n}$ also satisfies 1.1 (i),(ii) in $\prod H_{p,n}$ since G is a direct summand by [3], 55.5, 238 and consequently in $(\prod H_{p,n})_t$, which implies a contradiction since $\coprod H_{p,n}$ is basic in $(\prod H_{p,n})_t$. Now, if $G = G' \oplus W$, where $\coprod H_{p,n} \subset G'$ then 1.4 (i) implies $W = 0$. q.e.d.

References

[1] CORNER A.L.S.: Every countable reduced torsion-free ring is an endomorphism ring, Proc.London M.S. (3)13(1963),687.

[2] FUCHS L.: Abelian groups, Budapest 1958.

[3] FUCHS L.: Infinite abelian groups I, Acad.Press 1970.

[4] GRIFFITH P.A.: Infinite abelian group theory, The Univ.of Chicago Press 1970.

[5] KULIKOV L.: On the theory of abelian groups of arbitrary power, Mat.Sbornik 16(1945),129.

Matematicko-fyzikální fakulta

Karlova universita

Sokolovská 83

Československo

(Oblatum 27.11.1972)