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ON THE ALGEBRAIC CHARACTERIZATION OF SYSTEMS OF 1-1 PARTIAL MAPPINGS

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1. Let $\mathcal{X} = \{X_\alpha ; \alpha \in \mathcal{A}\}$ be a system of sets and $\mathcal{F} = \{f_a ; a \in A\}$ a system of certain subsets $f \subset X_\alpha \times X_\beta$ $(\alpha, \beta \in \mathcal{A})$. We can consider these subsets as multivalued partial mappings among sets of $\mathcal{X}$ which form the following operations on $\mathcal{F}$: a partial binary operation (the composition of relations $\circ : f, g \rightarrow f \circ g = \{(x, z) ; (x, y) \in g, (y, z) \in f\}$) and an unary one (the inverse relation $^{-1} : f \rightarrow f^{-1} = \{(x, y) ; (y, x) \in f\}$). $\mathcal{F}$ with these operations forms an algebra called an algebraization of the system of sets and relations.

On the other hand: We have an algebra $\underline{\mathcal{G}}$ with a partial binary operation $\cdot$ and a unary operation $^{-1}$ and we try to find a system of sets and partial mappings whose algebraization is the algebra $\underline{\mathcal{G}}$. We call such system of sets and mappings a representation of the algebra $\underline{\mathcal{G}}$.

It is well known that an algebraization of a system

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of mappings of a single set closed under the composition of mappings (the composition of mappings, the identity mapping and the inverse mapping, resp.) is a semigroup (group, resp.). Representations are given by the well-known Cayley's theorem. The problem of algebraizations and representations of categories has been solved by P. Freyd (see [3]). Similar representations of certain algebras are given in [4], too.

An algebraization of a system of all 1-1 partial mappings of a single set (including the empty mapping) is called an inverse semigroup; its representation was given in [1],[2]. In this paper, we solve a more general question of the algebraization of systems of 1-1 partial non-empty mappings closed under the inverse partial mappings and under the non-empty composition of partial mappings. (The exclusion of empty mappings is not substantial. We use it in order to simplify representations.) We give in this paper representations of algebras of 1-1 partial non-empty mappings among a set of sets, a class of sets, resp. (Theorem 1.2 resp.). In the second case we use the axiom of choice. In each of these cases we give a different representation. The correspondence between them is formulated in Theorem 3 - in fact, it is the matter of factorization.

2. Denote in this paper by $\underline{G} = (G, \cdot, ^{-1})$ an algebra on a class G, consisting of a partial binary operation $\cdot$ and a total unary operation $^{-1}$. Furthermore, if $X = \{X_\alpha ; \alpha \in A\}$ is a system of sets, F will always denote

a system of some non-empty 1-1 partial mappings among sets from X which is closed under the inverse partial mappings and under the composition of partial mappings.

Theorem 1. Let G, A be sets; let $\cdot$ ($^{-1}$ resp.) be a partial binary (total unary, resp.) operation on G. Then $\underline{G}$ is an algebraization of the system (X, F) if and only if the following conditions for any $a, b, c \in G$ hold:

(1) $(ab)c$ is defined if and only if $a(bc)$ is defined; then $(ab)c = a(bc)$;

(2) $(a^{-1})^{-1} = a$;

(3) ab is defined if and only if $b^{-1}a^{-1}$ is defined and then $(ab)^{-1} = b^{-1}a^{-1}$;

(4) $aa^{-1}a$ is defined and $aa^{-1}a = a$;

(5) $(aa^{-1})(bb^{-1}) = (bb^{-1})(aa^{-1})$ whenever one of these two expressions is defined.

Remark. It is easy to see that in the case of a total binary operation we obtain precisely the inverse semigroup axioms.

Proof of Theorem 1. Obviously, the algebraization of any system (X, F) satisfies conditions (1) - (5). On the other hand, from (1) - (5) for an algebra $\underline{G}$, further conditions follow:

(A) If ab is defined, then $a^{-1}ab$, abb^{-1} are also defined. ($(ab)^{-1}ab$, $ab(ab)^{-1}$ are defined (see (4)), hence from (3) and (1) $b^{-1}(a^{-1}ab)$, $(abb^{-1})a^{-1}$ are defined.)

(B) If we denote $J = \{aa^{-1};\ a \in G\}$, then for any $j_1, j_2, j_3 \in J$, equations $j_1 j_2 = j_2$, $j_2 j_3 = j_3$ imply

$j_1 j_3 = j_3$. (We have $j_3 = j_2 j_3 = (j_1 j_2) j_3 = j_1 (j_2 j_3) = j_1 j_3$.)

Denote R the following binary relation on G : for $a, b \in G$ there is $(a, b) \in R$ if and only if ab^{-1} is defined in G . Denote $\approx$ the equivalence generated by R . Now we can define the system (X, F) . Putting $X = \{X_\alpha ; \alpha \in A\}$, where X_α are just different classes of the equivalence $\approx$, we shall take for F a system of mappings f_a of sets X_α indexed by elements of G , where f_a are defined as follows:

for $x \in G$, $f_a(x)$ is defined if and only if xa is defined and $xaa^{-1} = x$. Then we put $f_a(x) = xa$. Clearly, $f_a(aa^{-1})$ is always defined. Moreover, $f_a(x) = f_a(y)$ implies $xa = ya$ and $xaa^{-1} = x$, $yaa^{-1} = y$, hence $x = y$. If $f_a(x)$, $f_a(y)$ are defined, we have $x \approx y$ and $f_a(x) \approx f_a(y)$. We can see that f_a are suitable non-empty one-to-one partial mappings and it remains to prove that (X, F) is a representation of G .

(a) For any $a, b \in G$ we have $f_b \circ f_a = f_{ab}$.
Whenever $f_{ab}(x)$ and $(f_b \circ f_a)(x)$ are defined, we find $f_b(f_a(x)) = (xa)b = x(ab) = f_{ab}(x)$. If $f_{ab}(x)$ is defined, we get $x(ab)(ab)^{-1} = x$. Thus xa is defined and from (B),(5), $(ab)(ab)^{-1}aa^{-1} = ab(ab)^{-1}$, $x^{-1}x(ab)(ab)^{-1} = x^{-1}x$ we can deduce $x^{-1}xaa^{-1} = x^{-1}x$. Thus $xaa^{-1} = x$, i.e. $f_a(x)$ is defined. Furthermore, $(xa)b$ is always defined and $xab \cdot b^{-1}a^{-1} = x$ implies that $xabb^{-1} = xa$, i.e. $(f_b \circ f_a)(x)$ is defined, too.

If $(f_b \circ f_a)(x)$ is defined, we have $xaa^{-1} = x$,

$xabb^{-1} = xa$. Thus $x(ab)(ab)^{-1} = xaa^{-1} = x$ and $f_{ab}(x)$ is also defined.

(b) For every $a \in G$, f_a and $f_{a^{-1}}$ are mutually inverse. We have $f_a \circ f_{a^{-1}} = f_{a^{-1}a}, f_{a^{-1}} \circ f_a = f_{aa^{-1}}$; $f_{a^{-1}a}(x) = xa^{-1}a = x$, $f_{aa^{-1}}(y) = yaa^{-1} = y$, whenever $f_{a^{-1}}(x)$, $f_a(y)$ are defined.

(c) For $a, b \in G$, $a \neq b$ implies $f_a \neq f_b$.

If $f_a = f_b$, then $f_a(aa^{-1}) = f_b(aa^{-1})$ and $f_a(bb^{-1}) = f_b(bb^{-1})$. Hence $aa^{-1} = aa^{-1}bb^{-1} = bb^{-1}aa^{-1} = bb^{-1}$ and $a = aa^{-1}b = bb^{-1}b = b$.

Theorem 2. Let G be a class, let X be a system of sets. Then an algebra $\underline{G}$ is the algebraization of a system (X, F) if and only if $\underline{G}$ satisfies:

(1) - (5) from Theorem 1;

(6) if we put $\tilde{G} = \{x \in G ; xx^{-1} = x^{-1}x\}$ and define $a \approx b$ if and only if there exist $a_0, \dots, a_n \in \tilde{G}$, $a_0 = a$, $a_m = b$ such that $a_i a_{i+1}$ is defined for $i = 0, \dots, m-1$, then $\{x \in \tilde{G} ; x \approx a\}$ is a set for every $a \in \tilde{G}$.

It is evident that the algebraization of any system F satisfies conditions (1) - (6). The sufficiency will result from the following three lemmas.

Lemma 1. $\mathfrak{M}(a) = \{x \in G ; xx^{-1} = aa^{-1}$ and $x^{-1}x = a^{-1}a\}$ is a set for every $a \in G$.

Proof. $g(x) = a^{-1}x$ defines a mapping g from $\mathfrak{M}(a)$ into $\tilde{G}$. Obviously, $a^{-1}x$ is always defined and $g(x)[g(x)]^{-1} = a^{-1}xx^{-1}a = a^{-1}aa^{-1}a = x^{-1}xx^{-1}x =$

$= x^{-1}aa^{-1}x = [q(x)]^{-1} q(x)$. Moreover, q is injective (if $q(x) = q(y)$, then $aa^{-1}x = aa^{-1}y$; hence $xx^{-1}x = yy^{-1}y$ and $x = y$) and $q(x) \approx q(y)$ for any $x, y \in \mathcal{M}(a)$. Condition (6) finishes the proof.

Lemma 2. Denote $d(a) = aa^{-1}$, $r(a) = a^{-1}a$ for every $a \in G$. Then there is a mapping $K : G \longrightarrow G$ with the following properties:

(I) $d[K(a)] = r(a)$, $r[K(a)] = d(a)$ for every $a \in G$;

(II) $[K(a)]^{-1} = K(a^{-1})$ for every $a \in G$;

(III) if $r(a) = d(b)$, then $K(b)K(a)$ is defined and $K(ab) = K(b) \cdot K(a)$;

(IV) if $r(a) = r(b)$ and $d(a) = d(b)$, then $K(a) = K(b)$.

Proof. We denote $J = \{aa^{-1}; a \in G\}$. We can define $(a,b) \in S$ if and only if $r(a) = d(b)$ and denote $\sim$ the equivalence generated by the binary relation S . Now, we can consider only classes of this equivalence. Let C be such a class and let $a \in C$. In view of $aa^{-1} \in J \cap C$, the class $J \cap C$ is non-empty and we can define $x_C \in J \cap C$ using the axiom of choice. $M(x) = \{a \in G; d(a) = x_C, r(a) = x\}$ is a non-empty set for every $x \in J \cap C$ according to the definition of $\sim$ and Lemma 1; so we can select $\overline{x} \in M(x)$. Now, we put $K(a) = [\overline{r(a)}]^{-1} \overline{d(a)}$ for every $a \in G$. If $a \in G$, then $d(\overline{r(a)})^{-1} = r(a)$, $r(\overline{r(a)})^{-1} = d(\overline{d(a)}) = x_C$, $r(\overline{d(a)}) = d(a)$, $r(K(a)) = d(a)$, $d(K(a)) =$

$$= (\overline{r(a)})^{-1}\overline{d(a)}(\overline{d(a)})^{-1}\overline{r(a)} = (\overline{r(a)})^{-1} d(\overline{r(a)})\,\overline{r(a)} = r(a).$$

Thus the definition of K is correct and (I) is proved. From $d(a) = r(a^{-1})$, $r(a) = d(a^{-1})$ it follows (II). If $r(a) = d(b)$, then $r(ab) = r(b)$, $d(ab) = d(b)$ and

$$K(ab) = (\overline{r(b)})^{-1}\overline{d(a)} = (\overline{r(b)})^{-1}\overline{d(b)}(\overline{d(b)})^{-1}\overline{d(a)} = K(b)K(a),$$

i.e. (III) holds. Obviously, (IV) holds, too.

Lemma 3. Let K be a mapping from Lemma 2. Then for every $a \in G$ $K(aa^{-1}) = aa^{-1}$.

Proof. $K(aa^{-1}) = K(a^{-1})K(a) = r[K(a)] = d(a) = aa^{-1}$.

Now, we can prove Theorem 2. The relation $\approx$ from (6) is clearly an equivalence. We can define the system (X, F) in this way: X is a system of all the classes of the equivalence $\approx$ (which are sets according to (6)). F is a system of all the mappings $\tilde{f}_a$ ($a \in G$) defined $\tilde{f}_a(x) = K(xa)\, xa$ whenever $xaa^{-1} = x$.

If $\tilde{f}_a(x)$, $\tilde{f}_a(y)$ are defined, then $x \approx y$ ($a_0 = x$, $a_1 = aa^{-1}$, $a_2 = y$, $a_i a_{i+1}$ are defined for $i = 0, 1$); we have also $\tilde{f}_a(x) \approx \tilde{f}_a(y)$ ($a_0' = \tilde{f}_a(x)$, $a_1' = a^{-1}a$, $a_2' = \tilde{f}_a(y)$). Obviously $\tilde{f}_a(aa^{-1})$ is always defined. If $\tilde{f}_a(x) = \tilde{f}_a(y)$, then $K(xa)xaa^{-1} = K(ya)yaa^{-1}$, i.e. $K(xa)x = K(ya)y$. From Lemma 3 it follows that $d(a^{-1}x^{-1}x) = d(a^{-1}y^{-1}y)$ and $r(a^{-1}x^{-1}x) = r(a^{-1}y^{-1}y)$. This fact implies $r(xa) = r(ya)$, $r(x) = r(y) = d(y) = d(x)$ and $d(xa) = d(ya)$. Then $K(xa) = K(ya)$, $K(a^{-1}x^{-1})K(xa)x = K(a^{-1}y^{-1})K(ya)y$,

$K(xaa^{-1}x^{-1})x = K(yaa^{-1}y^{-1})y$ and $x = y$. Thus $\tilde{f}_a$ are suitable non-empty one-to-one partial mappings.

Now we prove that $\tilde{f}_b \circ \tilde{f}_a = \tilde{f}_{ab}$ for any $a, b \in G$. If $(\tilde{f}_b \circ \tilde{f}_a)(x)$ is defined, then $K(a^{-1}x^{-1})K(xa)xabb^{-1} =$ $= K(a^{-1}x^{-1})K(xa)xa$ and from Lemmas 2 and 3 it follows that $xabb^{-1}a^{-1} = xaa^{-1} = x$, i.e. $\tilde{f}_{ab}(x)$ is defined. If $\tilde{f}_{ab}(x)$ is defined, then $x = xabb^{-1}a^{-1} =$ $= x(x^{-1}x)(aa^{-1})abb^{-1}a^{-1} = x(aa^{-1})(x^{-1}x)abb^{-1}a^{-1} = x(aa^{-1})(x^{-1}x) =$ $= xaa^{-1}$, i.e. $\tilde{f}_a(x)$ is defined. Furthermore, $K(xa)xa = K(xa)xa(bb^{-1})(a^{-1}a) = K(xa)xa(a^{-1}a)(bb^{-1}) =$ $= K(xa)xabb^{-1}$, hence $(\tilde{f}_b \circ \tilde{f}_a)(x)$ is defined, too. Finally, $(\tilde{f}_b \circ \tilde{f}_a)(x) = K[K(xa)xab]K(xa)xab =$ $= K(a^{-1}x^{-1}xab)K(xa)xab = K(xab)xab = \tilde{f}_{ab}(x)$.

Obviously $\tilde{f}_{aa^{-1}}(x)$ is defined if and only if $\tilde{f}_a(x)$ is defined; we have $\tilde{f}_{aa^{-1}}(x) = K(xaa^{-1})xaa^{-1} = K(x)x = x$. A similar consideration shows that $\tilde{f}_{a^{-1}a}(y) = y$, if $\tilde{f}_{a^{-1}}(y)$ is defined. Thus $\tilde{f}_a$ and $\tilde{f}_{a^{-1}}$ are mutually inverse.

Finally we have to prove that $a \neq b$ implies $\tilde{f}_a \neq \tilde{f}_b$. Suppose $\tilde{f}_a = \tilde{f}_b$. In the same way as in the proof of Theorem 1 we can prove that $aa^{-1} = bb^{-1}$, which implies $K(aa^{-1}a)aa^{-1}a = K(bb^{-1}b)bb^{-1}b$, i.e. $K(a)a =$ $= K(b)b$ and $a^{-1}a = b^{-1}b$. Thus $K(a) = K(b)$ and $K(a^{-1})K(a)a = K(b^{-1})K(b)b$, i.e. $a = b$.

<u>Theorem 3</u>. Let $\langle G, \cdot, {}^{-1} \rangle$ be an algebra from Theo-

rem 1, i.e. let G be a set. Let $\mathcal{U}, \mathcal{Z}$ resp. be systems of sets and some of their partial mappings which are the represenations of the algebra $\underline{G}$ in the sense of Theorem 1, 2 resp. Then $\mathcal{Z}$ is a factorization of $\mathcal{U}$.

Proof. Let us put $\tilde{G} = \{a \in G; aa^{-1} = a^{-1}a\}$. For certain sets $\overline{\mathcal{U}}, \overline{\mathcal{Z}}$ we have $\mathcal{U} = \langle\{O_p; p \in \overline{\mathcal{U}}\}, \{f_a; a \in G\}\rangle$, $\mathcal{Z} = \langle\{O'_q; q \in \overline{\mathcal{Z}}\}, \{\tilde{f}_a; a \in \tilde{G}\}\rangle$. (The definition of sets $\overline{\mathcal{U}}, \overline{\mathcal{Z}}$ and of mappings f_a, $\tilde{f}_a$ follows clearly from Theorems 1 and 2.)

We define $h : G \longrightarrow \tilde{G}$ as $h(x) = K(x) \cdot x$ and we shall show that h is the required factorization.

(a) $h(x)(h(x))^{-1} = K(x)xx^{-1}[K(x)]^{-1} = K(x)K(xx^{-1})K(x^{-1}) = x^{-1}x = x^{-1}K(xx^{-1})x = (h(x))^{-1} \cdot h(x)$, i.e. $h(x) \in \tilde{G}$. Putting $\tilde{y} = K(y^{-1})y$ for $y \in \tilde{G}$, we get $h(\tilde{y}) = K(\tilde{y})\tilde{y} = K(y)K(y^{-1})y = y$.

(b) For every $p \in \overline{\mathcal{U}}$ there exists $q \in \overline{\mathcal{Z}}$ such that $h(O_p) \subset O'_q$. It is sufficient to prove that $(x, y) \in R$ implies $h(x) \approx h(y)$. If we denote $a_0 = h(x)$, $a_1 = y^{-1}K(y^{-1})$, $a_2 = h(y)$, we can easily see that $a_i a_{i+1}$ is defined for $i = 0, 1$, i.e. $h(x) \approx h(y)$.

(c) For every $q \in \overline{\mathcal{Z}}$ there exists $p \in \overline{\mathcal{U}}$ such that $h^{-1}(O'_q) \subset O_p$. We have to prove that for any $x, y \in \tilde{G}$ xy^{-1} is defined, whenever xy is defined. We have $xy = xyy^{-1}y = xy^{-1}yy$ and xy^{-1} is defined, too.

(d) Finally, if $f_a(x)$ is defined, then $\tilde{f}_a(h(x)) = h(f_a(x))$. We have $h(x)aa^{-1} = K(x)xaa^{-1} = K(x)x = h(x)$ and $\tilde{f}_a(h(x))$ is defined. Moreover, we get $\tilde{f}_a(h(x)) =$

$$= K(K(x)xa)K(x)xa = K(x^{-1}xa)K(x)xa = K(xa)xa = h(xa) = h(f_a(x))$$

and Theorem 3 is proved.

R e f e r e n c e s

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