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ON PROJECTIVE LIMITS OF PROBABILITY SPACES

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The aim of this paper is to correct some results in the interesting paper of H.L. Rao [3].

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1. Pure Probabilities

1.1. Definition (see [3], 4.1). Let $P : \mathcal{A} \rightarrow [0, 1]$ be a finitely additive set function on an algebra $\mathcal{A} \subset \exp X$.

A ring $\mathcal{R} \subset \mathcal{A}$ is called P -pure if

(i) $A_m \in \mathcal{R}$ for $m \in \mathbb{N}$ (\mathbb{N} is the set of all non-negative integers), $A_m \supseteq \emptyset$ imply $P[A_{m_0}] = 0$ for some m_0 ,

(ii) $P[A] = \inf \{ \sum_{m \in \mathbb{N}} P[A_m] \mid A_m \in \mathcal{R} \text{ and } \bigcup_{m \in \mathbb{N}} A_m \supseteq A \}$

for each $A \in \mathcal{A}$.

If there exists a P -pure ring then P is said to be pure.

Remark. Any pure P is σ -additive ([3], 4.2) but the converse is not true as it will be shown below (beforehand, David Preiss constructed another counter-example).

1.2. Lemma (cf. [2], 7(ii)). Let $P : \mathcal{A} \rightarrow [0, 1]$ be a non-atomic probability, let $\mathcal{R} \subset \mathcal{A}$ be a P -pure ring, $E \in \mathcal{R}$,

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$P[E] > 0$. Then there exist $E_1, E_2 \in \mathcal{R}$ such that
 $E_1 \cup E_2 \subset E, E_1 \cap E_2 = \emptyset$ and $\frac{1}{4}P[E] > P[E_i] > 0$ for
 $i = 1, 2$.

Proof. As P is non-atomic there are $A_1, A_2 \in \mathcal{A}$
such that $A_1 \cup A_2 \subset E, A_1 \cap A_2 = \emptyset, P[A_1] = P[A_2] = \frac{1}{8}P[E]$.
There exist $B_i^j \in \mathcal{R} (i = 1, 2; j \in \mathbb{N})$ such that $\bigcup_{j \in \mathbb{N}} B_i^j \supset A_i$
and $P[\bigcup_{j \in \mathbb{N}} B_i^j] < \frac{1}{4}P[E]$ for $i = 1, 2$. Obviously
 $P[B_2^k \cap E] > 0$ for some $k \in \mathbb{N}$. As $\bigcup_{j \in \mathbb{N}} (B_1^j \cap E) \setminus B_2^k \supset$
 $\supset A_1 \setminus \bigcup_{j \in \mathbb{N}} B_2^j = A_1 \setminus [\bigcup_{j \in \mathbb{N}} B_2^j \setminus A_2]$ and $P[A_1] = \frac{1}{8}P[E],$
 $P[\bigcup_{j \in \mathbb{N}} B_2^j \setminus A_2] < \frac{1}{8}P[E]$ one has $P[\bigcup_{j \in \mathbb{N}} (B_1^j \cap E) \setminus B_2^k] > 0$.
Hence $P[(B_1^l \cap E) \setminus B_2^k] > 0$ for some $l \in \mathbb{N}$. The sets
 $E_1 = (B_1^l \cap E) \setminus B_2^k$ and $E_2 = B_2^k \cap E$ have the required
properties.

1.3. Proposition (cf. [2], 7(iii)). Let $P: \mathcal{A} \rightarrow [0, 1]$
be a non-atomic probability (on a σ -algebra \mathcal{A}) and let
 $\mathcal{R} \subset \mathcal{A}$ be a P -pure ring, $E \in \mathcal{R}, P[E] > 0$.
Then there exists $A \in \mathcal{R}$ such that $A \subset E, \text{card } A \geq \exp \kappa_0$
and $P[A] = 0$.

Proof will be only sketched here (it is essentially
the same as the proof of 7(iii) in [2]): by means of Lemma
1.2 one can (inductively) construct the sets
 $E(a_0, a_1, \dots, a_m), m \in \mathbb{N}, a_i = 0, 1$ for $i = 0, 1, \dots, m,$
such that $P[E(a_0, a_1, \dots, a_m)] > 0,$
 $E(a_0, a_1, \dots, a_m, 0) \cap E(a_0, a_1, \dots, a_m, 1) = \emptyset,$
 $E \supset E(a_0, a_1, \dots, a_m) \supset E(a_0, a_1, \dots, a_{m+1}),$

and put $A = \bigcap_{n \in \mathbb{N}} E_n$ where $E_n = \cup \{E(a_0, a_1, \dots, a_m) \mid a_i = 0, 1 \text{ for } 0 \leq i \leq m\}$.

Remarks. Sierpiński proved (supposing continuum-hypothesis) that there exists a non-atomic probability space all null-sets of which are at most countable (see e.g. [4]); such a probability is not pure due to 1.3 (cf. [2], 7(iv)). The properties of pure probabilities are very similar to those of compact ones (for definition of compact measure see [2]), e.g. indirect product of pure probabilities is pure. It is even pretty possible that these two notions (compact, pure) are not really distinct; this is the case for countably-generated (in the sense of Carathéodory) probabilities; the proofs will soon be published.

2. Projective Limits

M.M. Rao gave conditions for σ -additivity of projective limits in terms of extensions of given probabilities ([3], 4.5 - 4.7). However, some of them are not correctly formulated (see 2.3).

2.0. Notations. Below, D is a set directed by the relation \leq (i.e. $R \circ R = R$, $R \cap R^{-1} = \text{diagonal}$, $R \circ R^{-1} = D \times D$ where $R \subset D \times D$ realizes \leq), $\{\mathcal{F}_\alpha\}_{\alpha \in D}$ is a family of σ -algebras $\subset \text{exp } X$ such that $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$ for $\alpha \leq \beta$; $\mathcal{F} = \bigcup_{\alpha \in D} \mathcal{F}_\alpha$, $\sigma \mathcal{F}$ is the σ -algebra generated by \mathcal{F} . Given probabilities $P_\alpha: \mathcal{F}_\alpha \rightarrow [0, 1]$ for $\alpha \in D$ such that $P_\alpha[E] = P_\beta[E]$ for $E \in \mathcal{F}_\alpha \cap \mathcal{F}_\beta$, $P: \mathcal{F} \rightarrow [0, 1]$ is the

finitely additive set function such that $P[E] = P_\alpha[E]$ for $E \in \mathcal{F}_\alpha$.

2.1. Proposition (see 2.0). The following conditions are equivalent:

- (i) P is σ -additive;
- (ii) for any $\alpha \in D$ there exists a probability $\overline{P}_\alpha: \sigma\mathcal{F} \rightarrow [0,1]$ that extends P_α and for every such extensions the following statement holds:
for every $A_n \in \mathcal{F} (n \in N), A_n \searrow \emptyset$ and $\epsilon > 0$ there are $\alpha_0 \in D, m_0 \in N$ such that $\overline{P}_\alpha[A_n] < \epsilon$ for $\alpha \geq \alpha_0, n \geq m_0$ (= mapping $\langle \alpha, n \rangle \mapsto \overline{P}_\alpha[A_n]$ is continuous on $D \times N$);
- (iii) for any $\alpha \in D$ there exists a probability $\overline{P}_\alpha: \sigma\mathcal{F} \rightarrow [0,1]$ that extends P_α and $\lim_{n \rightarrow \infty} (\sup_{\alpha \in D} \overline{P}_\alpha[A_n]) = 0$ for every $A_n \in \sigma\mathcal{F} (n \in N)$ with $A_n \searrow \emptyset$ (= mapping $n \mapsto \overline{P}_\alpha[A_n]$ is continuous on N uniformly for all $\alpha \in D$).

Proof. Implications (ii) \implies (i) and (iii) \implies (i) are immediate. (i) \implies (ii) and (i) \implies (iii): to show the existence of the required extensions one can use for \overline{P}_α the (unique) extension of P on $\sigma\mathcal{F}$. If \overline{P}_α 's are arbitrary extensions of P_α 's and $A_n \in \mathcal{F}, A_n \searrow \emptyset$ then $P[A_{m_0}] < \epsilon$ for some m_0 and $A_{m_0} \in \mathcal{F}_{\alpha_0}$ for some α_0 . Hence $\overline{P}_\alpha[A_{m_0}] = P_\alpha[A_{m_0}] = P[A_{m_0}] < \epsilon$ for $\alpha \geq \alpha_0$ and $\overline{P}_\alpha[A_n] \leq \overline{P}_\alpha[A_{m_0}]$ for $n \geq m_0$.

Remark. The condition in 2.1(iii) can be reformulated like this:

$\{\overline{P}_\alpha \mid \alpha \in D\} \subset ca(X, \sigma\mathcal{F})$ is weakly sequentially compact (see [1], IV.9.1) or like this:

\overline{P}_α 's are uniformly λ -continuous for some $\lambda \in ca(X, \sigma\mathcal{F})$ (see [1], IV.9.2). But these conditions need not hold for every family $\{\overline{P}_\alpha\}$ of extensions (see 2.3).

2.2. Proposition (see 2.0). Let $D = N$ (N naturally ordered). The following conditions are equivalent:

(i) P is σ -additive;

(iv) for any $k \in N$ there exists a probability $\overline{P}_k : \sigma\mathcal{F} \rightarrow [0, 1]$ that extends P_k and for every such extensions and for every $A_m \in \mathcal{F}$ ($m \in N$), $A_m \searrow \emptyset$ it holds $\lim_{m \rightarrow \infty} (\sup_{k \in N} \overline{P}_k[A_m]) = 0$ (= mapping $m \mapsto \overline{P}_k[A_m]$ is continuous on N uniformly for all $k \in N$);

(v) for any $k \in N$ there exists a probability $\overline{P}_k : \sigma\mathcal{F} \rightarrow [0, 1]$ that extends P_k and such that $\lim_{k \rightarrow \infty} \overline{P}_k[A]$ exists for any $A \in \sigma\mathcal{F}$ (= mapping $k \mapsto \overline{P}_k[A]$ is continuous on N for any A).

Proof. Implication (iv) \implies (i) is clear, implication (v) \implies (i) is the theorem of Nikodým (see [1], III.7.4). (i) \implies (iv) and (i) \implies (v): the existence of extensions \overline{P}_k is obvious as in the proof of 2.1.

Let \overline{P}_k 's be arbitrary extensions of P_k 's, $A_m \in \mathcal{F}$, $A_m \searrow \emptyset$, $\varepsilon > 0$. For some m_1 it holds $P[A_{m_1}] < \varepsilon$, for some k_1 it holds $A_{m_1} \in \mathcal{F}_{k_1}$, hence $\overline{P}_{k_1}[A_{m_1}] = P_{k_1}[A_{m_1}] = P[A_{m_1}] < \varepsilon$ for $k \geq k_1$. For $i = 0, 1, \dots, k_1 - 1$

there are l_i such that $\overline{P}_i[A_{l_i}] < \epsilon$; put $n_0 = \max\{n_1, l_0, l_1, \dots, l_{n_1-1}\}$; then $\overline{P}_n[A_{n_0}] < \epsilon$ for any $n \in \mathbb{N}$.

2.3. Examples. (a) The condition in 4.5 of [3] does not necessarily hold for arbitrary extensions \overline{P}_α : Lebesgue probability on $[0,1]$ is the projective limit of all its restrictions to finite subalgebras and any such restriction can be extended as convex combination of Dirac measures. The family $\{\overline{P}_\alpha\}$ containing all these extensions works very wildly and does not satisfy any expected condition.

(b) This example shows (for $D = \mathbb{N}$) that a family $\{\overline{P}_n\}$ of extensions need not be terminally uniformly λ -continuous for any finite measure λ on \mathcal{F} :

For $n \in \mathbb{N}$, $\mathcal{F}_n \subset \exp[0,1]$ is the algebra of all the finite unions of intervals with end-points $\frac{k}{2^n}$, $k = 0, 1, \dots, 2^n$, P_n is the restriction of the Lebesgue probability on $[0,1]$ to \mathcal{F}_n , $\overline{P}_n = \frac{1}{2^n} \sum_{k=0}^{2^n} \delta_{x(n, k)}$ where $x(n, k) = \frac{2k-1}{2^{n+1}}$ and δ_x is the Dirac measure supported by x .

R e f e r e n c e s

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