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A NOTE ON VOLTERRA INTEGRAL EQUATIONS WITH DEGENERATE  
KERNEL

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In the paper several relations between the linear  
vector - valued Volterra integral equation

$$(I) \quad x(t) = a(t) + \int_0^t B(t, \tau) x(\tau) d\tau$$

and the initial - value problem

$$(D) \quad \begin{aligned} \dot{x} - P(t)x &= q(t), \\ x(0) &= x_0 \end{aligned}$$

are investigated. Particularly it is shown that under some weak assumptions the following three assertions are equivalent:

- (i) the kernel  $B$  of the equation (I) is degenerate;
- (ii) there exists a matrix  $P(t)$  such that the function  $B(., \tau)$  satisfies the equation in (D) with  $q = 0$ ;
- (iii) the solution of the equation (I) satisfies some special initial - value problem of the type (D).

Analogous results are obtained for the case of an

initial - value problem for a differential equation of a higher order.

The results generalize those obtained by J. Nagy and E. Nováková in [2] for a special type of the kernel  $B$ .

1. Notation. Let for  $m, n = 1, 2, \dots$   $R^{m \times n}$  ( $K^{m \times n}$ ) denote the space of all real (complex) matrices of the type  $m \times n$ . The  $m$ -dimensional vectors will be identified with the column matrices (of the type  $m \times 1$ ) for  $m = 1, 2, \dots$ , and  $R^m, K^m$  will stand for  $R^{m \times 1}, K^{m \times 1}$  respectively. Analogously for vector valued functions. We shall denote the identity matrices by  $I$  and the zero matrices by  $O$ .

Let  $G \subset R^n$  be a domain in  $R^n$ , let  $\bar{G}$  be the closure of  $G$ . Then  $C_{m \times n}^{(k)}(G)$  for  $m, n = 1, 2, \dots$ ;  $k = 0, 1, 2, \dots$  denotes the space of all  $m \times n$  complex  $k$ -times continuously differentiable matrix-valued functions on  $\bar{G}$ . (The function is  $0$ -times continuously differentiable if it is continuous; we define the  $0$ -th derivative of a given function to be equal to the function itself.)

Let  $m_i > 0, n_j > 0$  for  $i = 1, 2, \dots, p$ ;  $j = 1, 2, \dots, q$  be integers,  $Y_{ij} \in K^{m_i \times n_j}$ . We shall identify the matrix

$$\begin{bmatrix} Y_{11} & Y_{12} & \dots & Y_{1q} \\ Y_{21} & Y_{22} & \dots & Y_{2q} \\ \dots & \dots & \dots & \dots \\ Y_{p1} & Y_{p2} & \dots & Y_{pq} \end{bmatrix}$$

with the corresponding element of  $K^{M \times N}$ , where  $M = m_1 + m_2 + \dots + m_p$ ,  $N = n_1 + n_2 + \dots + n_q$ .

The partial derivatives of a function  $f$  with the domain in  $R^n$  will be denoted by

$$D^i f(u) = D^{i_1, \dots, i_p} f(u) = \frac{\partial^{i_1 + \dots + i_p} f(u_1, \dots, u_p)}{\partial u_1^{i_1} \dots \partial u_p^{i_p}}$$

where  $i = (i_1, \dots, i_p)$  denotes some multiindex,  $p = 1, 2, \dots$ . Further, the set  $\{[t, b] \in R^2 : t \geq b \geq 0\}$  will be denoted by  $\Delta$  and the interval  $(0, \infty)$  by  $R_+$ . Finally, in what follows, the symbols  $m, n$  will stand for integers,  $m \geq 1, n \geq 1, k \geq 0$  and  $P, B$  will be elements of  $C_{m \times n}^{(0)}(R_+), C_{m \times n}^{(0)}(\Delta)$  respectively.

2. Problem. The main purpose of the paper is to find some assumptions on the kernel  $B$  and the forcing function  $a$  so that the solution of the Volterra integral equation

$$(I) \quad x(t) = a(t) + \int_0^t B(t, s) x(s) ds, \quad t \geq 0$$

may satisfy some special initial value problem for an ordinary differential equation.

The following theorem is well known.

3. Theorem. Let  $P \in C_{m \times m}^{(0)}(R_+), q \in C_{m \times m}^{(0)}(R_+), b \geq 0, x_b \in K^{n \times m}$ . Then there exists a unique solution  $x \in C_{m \times m}^{(1)}(b, \infty)$  of the initial value problem

$$(3.1) \quad \dot{x} - P(t)x = q(t), \quad t > b,$$

$$(D) (3.2) \quad x(b) = x_b.$$

4. Remark. The following well known variation of constants formula:

$$(4.1) \quad x(t) = H(t)H(b)^{-1}x_b + \int_b^t H(t)H(u)^{-1}q(u)du, \quad t \geq b,$$

holds for the solution  $x$  of (D) where  $H \in C^{(1)}_{m \times m}(\mathbb{R}_+)$  is the solution of the square-matrix initial value problem ( $K$  is a regular square matrix)

$$(4.2) \quad \dot{X} = P(t)X, \quad X(0) = K.$$

The following theorem holds for the equation (I). (See R.K. Miller [1].)

5. Theorem. Let  $a \in C^{(h)}_{m \times m}(\mathbb{R}_+)$ ,  $B \in C^{(h)}_{m \times m}(\Delta)$ .

Then there exists a unique solution  $x \in C^{(h)}_{m \times m}(\mathbb{R}_+)$  of the equation (I), which is given by

$$(5.1) \quad x(t) = a(t) + \int_0^t R(t,s)a(s)ds, \quad t \geq 0,$$

where  $R$  is the resolvent kernel of the kernel  $B$ . This kernel  $R$  is the unique solution of the resolvent equation

$$(5.2) \quad R(t,s) = B(t,s) + \int_0^t B(t,u)R(u,s)du, \quad 0 \leq s \leq t.$$

6. Remark. In what follows we shall be interested especially in the case of degenerate kernels, i.e. kernels  $B$  of the form

$$(6.1) \quad B(t,s) = [v_{ij}(t,s)]_{i,j=1}^m$$

with  $v_{ij}(t,s) = u_{ij}(t)v_{ij}(s)$ ;  $t \geq s \geq 0$ ;  $i, j = 1, 2, \dots, m$ ;  $u_{ij}$ ,  $v_{ij}$  being some sufficiently many times continuously

differentiable matrix functions of the type  $1 \times k_{ij}$ ,  $k_{ij} \times 1$  respectively, defined on  $R_+$ .

7. Lemma. Let  $B \in C_{m \times m}^{(k)}(\Delta)$  be the degenerate kernel (6,1). Then there exist an integer  $m \geq 1$  and  $U \in C_{m \times m}^{(k)}(R_+)$ ,  $V \in C_{m \times m}^{(k)}(R_+)$  so that

$$(7.1) \quad B(t, s) = U(t)V(s), \quad t \geq s \geq 0.$$

It is possible to choose  $m \geq n$  and the matrix  $U$  in the form

$$U = [I, U_1].$$

Proof. Obviously, we can choose  $U$  in the form

$$U = \begin{bmatrix} \mu_{11} & \mu_{12} & \dots & \mu_{1m} & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \mu_{21} & \mu_{22} & \dots & \mu_{2m} & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & \mu_{m1} & \mu_{m2} & \dots & \mu_{mm} \end{bmatrix}$$

and the transposed matrix  $V^T$  of the matrix  $V$  in the form

$$V^T = \begin{bmatrix} v_{11}^T & 0 & \dots & 0 & v_{21}^T & 0 & \dots & 0 & \dots & v_{m1}^T & 0 & \dots & 0 \\ 0 & v_{12}^T & \dots & 0 & 0 & v_{22}^T & \dots & 0 & \dots & 0 & v_{m2}^T & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & v_{1m}^T & 0 & 0 & \dots & v_{2m}^T & \dots & 0 & 0 & \dots & v_{mm}^T \end{bmatrix}.$$

So we obtain

$$m = n + \sum_{i,j=1}^n k_{ij}.$$

8. Theorem. Let  $B \in C_{m \times m}^{(\mathbb{R}_+)}(\Delta)$  be the degenerate kernel (6.1),  $a \in C_m^{(0)}(\mathbb{R}_+)$ . Then there exist an integer  $\mu, \mu > m$  and  $\tilde{U}, \tilde{V} \in C_{\mu \times \mu}^{(\mathbb{R}_+)}(\mathbb{R}_+)$  such that  $U(t)$  is a regular square-matrix for all  $t \geq 0$  and the following assertion holds:

Let us define  $\tilde{B} \in C_{\mu \times \mu}^{(\mathbb{R}_+)}(\Delta)$ ,  $\tilde{a} \in C_\mu^{(0)}(\mathbb{R}_+)$  by means of

$$(8.1) \quad \tilde{B}(t, \tau) = \tilde{U}(t) \tilde{V}(\tau), \quad t \geq \tau \geq 0, \\ \tilde{a} = \begin{bmatrix} a \\ 0 \end{bmatrix}.$$

Let  $\tilde{x} \in C_\mu^{(0)}(\mathbb{R}_+)$ ,  $x \in C_m^{(0)}(\mathbb{R}_+)$ ,  $y \in C_{\mu-m}^{(0)}(\mathbb{R}_+)$  and let it hold

$$\tilde{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Then

(i) if  $\tilde{x}$  is a solution of the equation

$$(8.2) \quad \tilde{x}(t) = \tilde{a}(t) + \int_0^t \tilde{B}(t, \tau) \tilde{x}(\tau) d\tau, \quad t \geq 0,$$

then  $x$  is a solution of (I);

(ii) if  $x$  is a solution of (I) then there exists such a  $y \in C_{\mu-m}^{(0)}(\mathbb{R}_+)$  that  $\tilde{x} = \begin{bmatrix} x \\ y \end{bmatrix}$  is a solution of (8.2).

Proof. We can put

$$\tilde{U} = \begin{bmatrix} U & I \\ I & 0 \end{bmatrix} \in C_{(m+m) \times (m+m)}^{(\mathbb{R}_+)}(\mathbb{R}_+),$$

$$\tilde{V} = \begin{bmatrix} V & 0 \\ 0 & 0 \end{bmatrix} \in C_{(m+m) \times (m+m)}^{(\mathbb{R}_+)}(\mathbb{R}_+),$$

where  $U, V$  are matrices of the types  $m \times m, m \times m$  respectively described in Lemma 7. Then  $\mu = m + m$ ,

$$\tilde{\mathbf{B}}(t, s) = \begin{bmatrix} \mathbf{B}(t, s) & 0 \\ \mathbf{V}(s) & 0 \end{bmatrix}$$

and (i) holds. Let  $x$  be the solution of (I). Set

$$y(t) = \int_0^t \mathbf{V}(s)x(s)ds, \quad t \geq 0.$$

Then  $\tilde{x}$  satisfies (8.2) and (ii) holds as well.

9. Remark. For some special kernels  $\mathbf{B}$  the conclusion of Theorem 8 (or, more precisely, its easy modification) holds with  $n = m$ .

Lemma 7 asserts that each degenerate kernel may be expressed in the form (7.1). So we shall pay attention only to the degenerate kernels of this type. From Theorem 8 it follows that each equation (8.2) with a degenerate kernel may be complemented so that the equation (8.2) with the kernel (8.1) will be obtained. Therefore it is sufficient to consider only such equations (I) with a degenerate kernel where the kernel  $\mathbf{B}$  is of the form (7.1) with a regular square matrix  $\mathbf{U}$ .

10. Theorem. Let  $\mathbf{U} \in C_{m \times m}^{(0)}(\mathbb{R}_+)$ ,  $\mathbf{V} \in C_{m \times m}^{(0)}(\mathbb{R}_+)$ ,  $\mathbf{B}(t, s) = \mathbf{U}(t)\mathbf{V}(s)$ ,  $t \geq s \geq 0$  and let  $\mathbf{E} \in C_{m \times m}^{(1)}(\mathbb{R}_+)$  be the solution of the matrix initial value problem

$$(10.1) \quad \dot{\mathbf{E}} = \mathbf{V}(t)\mathbf{U}(t)\mathbf{E}, \quad \mathbf{E}(0) = \mathbf{I}.$$

Then the function

$$(10.2) \quad \mathbf{R}(t, s) = \mathbf{U}(t)\mathbf{E}(t)\mathbf{E}(s)^{-1}\mathbf{V}(s), \quad t \geq s \geq 0$$

is the resolvent kernel of the kernel  $\mathbf{B}$ .



Proof. Clearly

$$\begin{aligned} \int_{\rho}^t \mathbf{B}(t, \mu) \mathbf{R}(\mu, \rho) d\mu &= \int_{\rho}^t \mathbf{U}(t) \mathbf{V}(\mu) \mathbf{U}(\mu) \mathbf{E}(\mu) \mathbf{E}(\rho)^{-1} \mathbf{V}(\rho) d\mu = \\ &= \mathbf{U}(t) \int_{\rho}^t \mathbf{V}(\mu) \mathbf{U}(\mu) \mathbf{E}(\mu) d\mu \mathbf{E}(\rho)^{-1} \mathbf{V}(\rho) = \mathbf{U}(t) \int_{\rho}^t \dot{\mathbf{E}}(\mu) d\mu \mathbf{E}(\rho)^{-1} \mathbf{V}(\rho) = \\ &= \mathbf{U}(t) [\mathbf{E}(t) - \mathbf{E}(\rho)] \mathbf{E}(\rho)^{-1} \mathbf{V}(\rho) = \mathbf{R}(t, \rho) - \mathbf{B}(t, \rho), \\ t &\geq \rho \geq 0, \end{aligned}$$

so that  $\mathbf{R}$  satisfies the resolvent equation (5.2).

11. Theorem. Let  $\mathbf{B} \in \mathbf{C}_{m \times m}^{(1)}(\Delta)$ .

Then the following three assertions are equivalent:

(i) there exist  $\mathbf{U} \in \mathbf{C}_{m \times m}^{(1)}(\mathbb{R}_+)$  regular on  $\mathbb{R}_+$  and  $\mathbf{V} \in \mathbf{C}_{m \times m}^{(1)}(\mathbb{R}_+)$  so that

$$(11.1) \quad \mathbf{B}(t, \rho) = \mathbf{U}(t) \mathbf{V}(\rho), \quad t \geq \rho \geq 0;$$

(ii) there exists  $\mathbf{P} \in \mathbf{C}_{m \times m}^{(0)}(\mathbb{R}_+)$  (which is uniquely determined by  $\mathbf{B}$ ) so that

$$(11.2) \quad \mathcal{D}^{1,0} \mathbf{B}(t, \rho) - \mathbf{P}(t) \mathbf{B}(t, \rho) = 0, \quad t \geq \rho \geq 0;$$

(iii) there exists  $\mathbf{P} \in \mathbf{C}_{m \times m}^{(0)}(\mathbb{R}_+)$  (which is uniquely determined by  $\mathbf{B}$ ) so that for all  $\mathbf{a} \in \mathbf{C}_m^{(1)}(\mathbb{R}_+)$  the solution  $\mathbf{x}$  of the equation (I) satisfies the initial value problem

$$(11.3) \quad \dot{\mathbf{x}} - [\mathbf{P}(t) + \mathbf{B}(t, t)] \mathbf{x} = \dot{\mathbf{a}}(t) - \mathbf{P}(t) \mathbf{a}(t),$$

$$(11.4) \quad \mathbf{x}(0) = \mathbf{a}(0)$$

(This  $\mathbf{P}$  is the same as that in (ii).)

Proof. (i)  $\Rightarrow$  (ii) From (11.1) it follows

$$\mathcal{D}^{1,0} \mathbf{B}(t, \rho) = \dot{\mathbf{U}}(t) \mathbf{V}(\rho) = \dot{\mathbf{U}}(t) \mathbf{U}(t)^{-1} \mathbf{U}(t) \mathbf{V}(\rho) = \mathbf{P}(t) \mathbf{B}(t, \rho)$$

where  $P(t) = \dot{U}(t)U(t)^{-1}$ ,  $t \geq 0$ .

(ii)  $\implies$  (i) Let  $H$  be the fundamental matrix of the system  $\dot{x} = P(t)x$ . Then (11.2) and Theorem 4 imply (11.1), where  $U(s) = H(s)$ ,  $V(s) = H(s)^{-1}B(s, s)$ ,  $s \geq 0$ .

(ii)  $\implies$  (iii) Let  $x$  be the solution of (I);  $a$ ,  $B$  continuously differentiable. Then

$$\dot{x}(t) = \dot{a}(t) + B(t, t)x(t) + \int_0^t D^{1,0}B(t, s)x(s)ds, \quad t \geq 0.$$

Simple calculation gives

$$(11.5) \quad \dot{x}(t) - [P(t) + B(t, t)]x(t) = \int_0^t [D^{1,0}B(t, s) - P(t)B(t, s)]x(s)ds + \dot{a}(t) - P(t)a(t), \quad t \geq 0.$$

Now (11.3) follows from (11.2) and (11.5).

(iii)  $\implies$  (ii) Let the solution  $x$  of (I) satisfy the initial value problem (11.3-4). Then (11.5) holds. Hence and from (11.3) we obtain

$$(11.6) \quad \int_0^t [D^{1,0}B(t, s) - P(t)B(t, s)]x(s)ds = 0, \quad t \geq 0.$$

From the equation (I) it follows that for each  $x \in C_m^{(1)}(\mathbb{R}_+)$  there exists  $a \in C_m^{(1)}(\mathbb{R}_+)$  so that  $x$  is the solution of (I). So (11.6) holds for all  $x \in C_m^{(1)}(\mathbb{R}_+)$  and (11.2) is satisfied.

12. Remark. Theorems 8 and 11 imply the following assertion for a degenerate kernel  $B \in C_{m \times m}^{(1)}(\Delta)$  and  $a \in C_m^{(1)}(\mathbb{R}_+)$ . It is always possible to complement the matrix  $B$  and the forcing function  $a$  in (I) so that the new kernel satisfies the equation of the form

(11.2) and the solution of the complemented equation satisfies the initial value problem of the form (11.3-4).

It also follows from Theorems 7 and 10 that the resolvent kernel  $\mathbf{R}$  of a smooth degenerate kernel  $\mathbf{B}$  is given by (10.2).

13. Remark. Theorems 10 and 11 imply immediately: if a kernel  $\mathbf{B}$  fulfils the equation (11.2) then

- (i)  $\mathbf{B}$  is degenerate;
- (ii) a solution of (I) (with smooth  $\mathbf{a}$ ) is also a solution of (11.3-4);
- (iii) the resolvent kernel  $\mathbf{R}$  may be written in the form (10.2).

The investigations described above may be modified and generalized in many ways. One of such modifications will be described now.

14. Theorem. Let  $A_0, A_1, \dots, A_{n_0} \in C_{n \times n}^{(0)}(\mathbb{R}_+)$ ,  $A_{n_0} = I$ ,  $B \in C_{m \times m}^{(n_0)}(\Delta)$ ,  $\mathbf{a} \in C_m^{(n_0)}(\mathbb{R}_+)$ .

Let for all  $\rho \geq 0$  the function  $\mathbf{B}(\cdot, \rho)$  satisfy the equation

$$(14.0) \quad \sum_{i=0}^{n_0} A_i(t) D^i \mathbf{x}(t) = 0$$

on  $\langle \rho, \infty \rangle$ .

Then

- (i) the kernel  $\mathbf{B}$  is degenerate;
- (ii) the function  $\mathbf{x} \in C_m^{(n_0)}(\mathbb{R}_+)$  is a solution of (I) if and only if it is a solution of the initial value problem

$$(14.1) \quad \sum_{\ell=0}^{k_0} F_{\ell}(t) D^{\ell} x(t) = q(t), \quad t > 0,$$

$$(14.2) \quad D^i x(0) - \sum_{\ell=0}^{i-1} G_{i\ell}(0) D^{\ell} x(0) = D^i a(0); \quad i = 0, 1, \dots, k_0 - 1;$$

where

$$(14.3) \quad F_{\ell}(t) = A_{\ell}(t) - \sum_{\kappa=0}^{k_0-\ell-1} \binom{\kappa+\ell}{\ell} \sum_{j=0}^{k_0-\ell-\kappa-1} A_{\ell+\kappa+j+1}(t) \times \\ \times D^{\kappa} D^{j,0} B(t,t); \quad \ell = 0, 1, \dots, k_0;$$

$$(14.4) \quad q(t) = \sum_{i=0}^{k_0} A_i(t) D^i a(t); \quad t \geq 0,$$

$$(14.5) \quad G_{i\ell}(t) = \sum_{j=0}^{i-\ell-1} \binom{i-j-1}{\ell} D^{i-j-\ell-1} D^{j,0} B(t,t), \quad t \geq 0; \\ \ell = 0, 1, \dots, i-1; \quad i = 0, 1, \dots, k_0;$$

and where we set  $\sum_{i=0}^{\mu} \dots = 0$  whenever  $\mu < 0$ .

Proof. We prove the assertion (i). Let us introduce the matrix functions

$$\tilde{x} = \begin{bmatrix} x \\ \dot{x} \\ \vdots \\ x^{(k_0-1)} \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -A_0 & -A_1 & -A_2 & \dots & -A_{k_0-1} \end{bmatrix}.$$

Clearly  $\tilde{A} \in C_{k_0 m \times k_0 m}^{(0)}(\mathbb{R}_+)$  and  $x \in C_m^{(k_0)}(\mathbb{R}_+)$  satisfies the equation (14.1) if and only if  $\tilde{x} \in C_{k_0 m}^{(1)}(\mathbb{R}_+)$  and  $\tilde{x}$  is the solution

$$\dot{\tilde{x}} = \tilde{A}(t) \tilde{x}.$$

If  $\tilde{H}$  is the fundamental matrix of the last equation, then its solution  $\tilde{x}$  may be written in the form

$$\tilde{x}(t) = \tilde{H}(t)\tilde{x}(0), \quad t \geq 0.$$

Setting

$$\tilde{H} = \begin{bmatrix} H_0 \\ H_1 \end{bmatrix}$$

where the sub-matrix  $H_0 \in C_m^{(1)} \times C_m \times C_m(\mathbb{R}_+)$ , we get for the solution  $x$  of the equation (14.0)

$$x(t) = H_0(t)\tilde{x}(0), \quad t \geq 0.$$

Using this and the assumptions of the theorem we obtain

$$B(t, \rho) = H_0(t)\tilde{H}(\rho)^{-1} \begin{bmatrix} B(\rho, \rho) \\ D^{1,0}B(\rho, \rho) \\ \vdots \\ D^{\mu-1,0}B(\rho, \rho) \end{bmatrix}, \quad t \geq \rho \geq 0$$

so that  $B$  is degenerate and the assertion (i) holds.

Now we shall consider (ii). From (14.3) it follows  $F_{\rho_0} = A_{\rho_0} = I$ . The values  $x(0), Dx(0), \dots, D^{\mu-1}x(0)$  are uniquely defined by (14.2). It is possible to transform the equation (14.1) into the first order differential equation as we transformed the equation (14.0) above. Hence and from Theorem 3 it follows that the initial value problem (14.1-2) is uniquely solvable in  $C_m^{(\mu)}(\mathbb{R}_+)$ .

Let  $x$  be the solution of the equation (I). Let us express the derivatives  $D^i x$  in the form

$$(14.6) \quad D^i x(t) = D^i a(t) + \sigma^i x(t) + \int_0^t D^{i,0} B(t, \rho) x(\rho) d\rho;$$

$i = 0, 1, \dots, k$  . Differentiating the both sides of the equation (I) and using (14.5-6) we obtain

$$(14.7) \quad \sigma^i x(t) = \sum_{j=0}^{i-1} D^{i-j-1} [(D^{j,0} B(t,t))x(t)] = \\ = \sum_{j=0}^{i-1} G_{ij}(t) D^j x(t); \quad t \geq 0; \quad i = 0, 1, \dots, k;$$

$$(14.8) \quad \sum_{i=0}^k A_i(t) \sigma^i x(t) = \sum_{i=0}^k A_i(t) \sum_{j=0}^{i-1} G_{ij}(t) D^j x(t) = \\ = \sum_{j=0}^k [A_j(t) - F_j(t)] D^j x(t), \quad x \geq 0 .$$

One has

$$\sum_{i=0}^k A_i(t) D^i x(t) = \sum_{i=0}^k A_i(t) D^i a(t) + \sum_{i=0}^k A_i(t) \sigma^i x(t) + \\ + \int_0^t \sum_{i=0}^k A_i(t) D^{i,0} B(t,s) x(s) ds, \quad t \geq 0 .$$

Since  $B(\cdot, s)$  is the solution of (14.1), the last term equals zero. Hence using (14.8), we obtain (14.1), where  $F_j, Q$  are defined by means of (14.3-4) respectively. Putting  $t = 0$  in (14.6) and using (14.7) we obtain the initial conditions (14.2).

Conversely, since the solution of the initial value problem (14.1-2) is unique and the solution of (I) exists, it follows from the above argument that the solution of the initial value problem (14.1-2) solves (I).

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