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APPROXIMATION BY HILL FUNCTIONS II

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1. Introduction. The problem of the approximation in Sobolev spaces by piecewise smooth function plays a very important role in applications today. In [1] [see also [8]] we studied this problem for a special class of approximating functions. [2] deals with a related problem. [3] studies the problems very similar to those in [1]. There are other significant results in this field, see e.g. [4], [5], and [6] and others.

Another approach to the approximation problems by piecewise smooth functions is in [7].

Problems of the mentioned type play very important roles in the finite element method. See e.g. [8] - [15] and others.

This paper deals with the problems of approximation on less dimensional manifolds and simultaneous approximation on manifolds of different dimensions.

These questions are important in the application, in finite element methods etc.

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2. Some Notions. Through the entire paper R_m be the m-dimensional Euclidean space, $x = (x_1, \dots, x_m)$, $\|x\|^2 = \sum_{i=1}^{m} x_i^2$ and $dx = dx_1 \dots dx_m$.
Further let

$$R_m^+ = E[x ; x_m > 0] ,$$
$$R_m^- = E[x ; x_m < 0] .$$

Let $\Omega \subset R_m$ be a bounded region and $\Omega^\bullet$ its boundary. We assume that $\Omega^\bullet$ is an $(m-1)$-dimensional manifold. Mostly we will asume that $\Omega^\bullet$ is infinitely times differentiable and we write in this case $\Omega^\bullet \in C^\infty$.
For $h > 0$ let $\Omega_h^\bullet = E[x \in \Omega ; d(x, \Omega^\bullet) < h]$ and

$$\Omega_h = E[x \in \Omega ; d(x, \Omega^\bullet) > h] ,$$

where $d(x, \Omega^\bullet)$ means the distance from x to $\Omega^\bullet$.
Let $L_2(\Omega)$ (resp. $L_2(R_m)$) be the space of square integrable functions u on Ω (resp. R_m) such that

$$\|u\|^2_{L_2(\Omega)} = \int_\Omega |u|^2 dx < \infty .$$

Analogously we define $L_2(R_m)$. Sometimes we shall write $L_2(\Omega) = W_2^0(\Omega)$ (resp. $L_2(R_m) = W_2^0(R_m)$).

Let $\mathcal{E}(\bar{\Omega})$ (resp. $\mathcal{E}(R_m)$) be the space of all infinite times differentiable functions on $\bar{\Omega}$ (resp. R_m) and such that all derivatives are continuously prolongable on $\Omega^\bullet$.
Furthermore let $\mathcal{D}(\Omega) \subset \mathcal{E}(\bar{\Omega})$ (resp. $\mathcal{D}(R_m)$) be the subspace of all functions with compact support in Ω (resp. R_m). Let ℓ be an integer $\ell \geq 1$. The Sobolev space $W_2^\ell(\Omega)$ (resp. $W_2^\ell(R_m)$) will be the closure of $\mathcal{E}(\bar{\Omega})$ (resp. $\mathcal{D}(R_m)$) in the norm $\|\cdot\|_{W_2^\ell(\Omega)}$ (resp.

$\| \cdot \|_{W_2^\ell(R_m)}$) where

$$\| u \|^2_{W_2^\ell(\Omega)} = \sum_{0 \le |\alpha| \le \ell} \| D^\alpha u \|^2_{L_2(\Omega)}$$

where

$$D^\alpha = \frac{\partial^{\alpha_1 + \dots + \alpha_m}}{\partial x_1^{\alpha_1} \, \partial x_m^{\alpha_m}} \quad , \quad \alpha \equiv (\alpha_1, \dots, \alpha_m) \quad ,$$

$\alpha_i \ge 0$, $i = 1, \dots, m$, $|\alpha| = \sum_{i=1}^{m} \alpha_i$.

(Quite analogously we define $\| \cdot \|_{W_2^\ell(R_m)}$.)

Let $\mathring{W}_2^\ell(\Omega) \subset W_2^\ell(\Omega)$ be the closure of $\mathcal{D}(\Omega)$ (resp. $\mathcal{D}(R_m)$). Let $f \in \mathcal{E}(\overline{\Omega})$, then $f^\bullet$ be a function defined on $\Omega^\bullet$ such that for $x \in \Omega^\bullet$ we have $f^\bullet(x) = f(x)$. Later we shall also use this notation if $f \in W_2^\ell(\Omega)$ and $f^\bullet$ may be defined on $\Omega^\bullet$ in a sense of traces.

Let now $\Omega^\bullet \in C^\infty$ and let $\psi_i \in \mathcal{D}(R_m)$, $\psi_i(x) \ge 0$, $i_0 = 1, \dots, \nu$ be a system of functions such that

$$\sum_{i=1}^{\nu} \psi_i^\bullet = 1 \qquad \text{on } \Omega^\bullet .$$

Further let there be a system of local coordinates $x_i^{[s]}$, $i = 1, \dots, m$, $s = 1, \dots, \nu$ and $(m-1)$-dimensional domains $J_s \subset R_m$, $J_s \in C^\infty$ and functions φ_s defined on J_s such that there is one to one infinitely times differentiable mapping χ_s of J_s such that $\chi_s(J_s) = \Omega^\bullet_s$ where

$$\Omega^\bullet_s = E\,[(x_1^{[s]}, \dots, x_{m-1}^{[s]}, \varphi(x_1^{[s]}, \dots, x_{m-1}^{[s]})) ; (x_1^{[s]}, \dots, x_{m-1}^{[s]}) \in J_s]$$

and so that

$$E\,[x \in \Omega^\bullet ; \psi_s(x) > 0] \subset$$

$$\subset E\,[(x_1^{[s]}, \dots, x_{m-1}^{[s]}, \varphi(x_1^{[s]}, \dots, x_{m-1}^{[s]})) ; (x_1^{[s]}, \dots, x_{m-1}^{[s]}) \in (J_s)_H] ,$$

with $H > 0$.

It is easy to see that such a system of functions

ψ_i, φ_i and domains J_i really exists.
Let f be defined on Ω°. Then the function $f_s = \psi_s f = 0$ everywhere outside of Ω°_s.
Therefore the function $f_s(\chi_s(x))$ is defined on J_s and has compact support.

Let us introduce the Sobolev space on Ω°. Let ℓ be an integer $\ell \geq 0$. The Sobolev space $W_2^\ell(\Omega^\circ)$ with the norm $\|\cdot\|_{W_2^\ell(\Omega^\circ)}$ is the space of all functions f defined on Ω° and such that

$$\|f\|^2_{W_2^\ell(\Omega^\circ)} = \sum_{s=1}^{\nu} \|f_s\|^2_{W_2^\ell(J_s)} < \infty .$$

We defined the Sobolev space $W_2^\ell(\Omega^\circ)$ for $\ell \geq 0$, ℓ integral. For ℓ negative, ℓ integral we shall define the space as a dual one, namely for $\ell \geq 0$

$$W_2^{-\ell}(\Omega^\circ) = (W_2^\ell(\Omega^\circ))' .$$

More about that see [17], p.35.

We have introduced the Sobolev spaces with integral derivatives. It is possible to show that tne norm $\|\cdot\|_{W_2^\ell(R_m)}$ may be introduced in terms of Fourier transform also (up to equivalency). For Ω with $\Omega^\circ \in C^\infty$ [and R_m] we may construct Hilbert scales and get spaces with fractional derivatives. See [17] and [18].

The norms of these interpolated spaces $W_2^k(\Omega)$, $W_2^k(\Omega^\circ)$, $W_2^k(R_m)$ are equivalent with Aronszajn-Slobodetskij norm. See e.g.[18]. E.g. for $0 < \alpha = [\alpha] + \sigma$, $0 < \sigma < 1$ and $[\alpha]$ integral we may define

(2.1) $$\|u\|^2_{W_2^\alpha(\Omega)} = \|u\|^2_{W_2^{[\alpha]}(\Omega)} + \sum_{|k|=[\alpha]} \|D^k u\|^2_{W_2^\sigma(\Omega)}$$

where

$$(2.2)\quad \|u\|^2_{W_2^\sigma(\Omega)} = \int_\Omega\int_\Omega \frac{(u(t)-u(\tau))^2}{(t-\tau)^{n+2\sigma}}\,dt\,d\tau \quad .$$

Analogously to (2.1) and (2.2) we may define the norm $\|\cdot\|_{W_2^\alpha(\Omega^\circ)}$ which is equivalent with the interpolated norm.

By interpolation we may define also $\mathring{W}_2^\alpha(\Omega)$, $0 \leq \alpha$ for α non-integral. We have to underline here that for $\alpha =$ integer $+ \frac{1}{2}$ the space $\mathring{W}_2^\alpha(\Omega)$, obtained by interpolation for α integral is not an equivalent one with the closure of $\mathcal{D}(\Omega)$ in the norm (2.1). More about that see [17].

Let us state now a lemma.

Lemma 2.1. Let ℓ, k be real, $|\ell| \leq L$, $|k| \leq L$, $L < \infty$. Then there exist operators Λ_h, $0 < h < 1$ which map $W_2^\ell(\Omega^\circ)$ into $W_2^k(\Omega^\circ)$ so that 1)

$$(2.3)\quad \|\Lambda_h f\|_{W_2^k(\Omega^\circ)} \leq C\|f\|_{W_2^\ell(\Omega^\circ)}\, h^{-\mu} ,$$

where

$$(2.4)\qquad \mu = \min[0, \ell - k]$$

and for $k \leq \ell$

$$(2.5)\quad \|f - \Lambda_h f\|_{W_2^k(\Omega^\circ)} \leq C\|f\|_{W_2^\ell(\Omega^\circ)}\, h^{\ell-k}$$

and C does not depend on h and f (may depend on L and Ω).

Proof. Because of our definition of $W_2^\ell(\Omega^\circ)$ with $f_h(\chi_h(x))$ having compact support in J_h, it is enough to show the existence of such an operator which maps

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1) Through the paper C is a generic constant with different values on different places.

$W_2^{l}(R_{n-1})$ into $W_2^{k}(R_{n-1})$, which has the properties (2.3) - (2.5) and furthermore is such that if f has compact support S, then $\Lambda_h f$ has a support $S^{(h)}$, where

$$S^{(h)} \subset E[x \in R_{n-1} ; d(x,S) \leq \lambda h]$$

and $d(x,S)$ is the distance of x to S and λ does not depend on h and f.

Using the technique of Fourier transform such an operator may be easily constructed.
The operator has a form of a convolution and its properties may be checked up the same way as in Part 1 of the proof of the theorem 4.1 in [1].

A notion which plays a basic role is a (t, k)-regular system $\mathcal{V}_h^{t,k}(R_n)$, where $0 < h < 1$, $t \geq k \geq 0$.

A one parametric system of functions $g \in \mathcal{V}_h^{t,k}(R_n)$ defined for every $0 < h < 1$ is called a (t, k)-regular system $(t \geq k \geq 0)$ if and only if

(1) $\mathcal{V}_h^{t,k}(R_n) \subset W_2^{k}(R_n)$,

(2) if $f \in W_2^{l}(R_n)$ then there exists $g \in \mathcal{V}_h^{t,k}(R_n)$ such that for any $0 \leq s \leq k \leq l$

(2.6) $$\| g - f \|_{W_2^{s}(R_n)} \leq C h^{\mu} \| f \|_{W_2^{l}(R_n)}$$

where

$$\mu = \min(l - s, t - s)$$

and C does not depend on s, h and f.

(3) If $f \in W_2^{l}(R_n)$ has compact support S then g in (2.6) has compact support $S^{(h)}$,

$$S^{(h)} = E[x \in R_m ;\ d(x, S) \leq \lambda h]$$

where λ does not depend on h and f (depends only on k, ℓ).

The notion of (t, k)-regular system $\mathcal{V}_h^{t,k}(R_m)$ is a generalization of the system of the hill functions introduced in [1].

In [1] we defined the system $\mathcal{V}_h^{t,k}(R_m)$ as the totality of all the functions of the form

$$\sum_{j=1}^{r} \sum_{p} c(p, j)\, \omega_j\left(\frac{x}{h} - p\right) \tag{2.7}$$

where $p = (p_1, \dots, p_m)$, p_i integral and $\omega_j(x) \in W_2^k(R_m)$ are fixed functions with compact support.

In [1] we have studied sufficient conditions for $\omega_j(x)$ such that the system (2.6) be (t, k)-regular.

3. <u>Approximation by Hill Functions on Ω and $\Omega^\bullet$</u>.

We studied in [1] the approximation problems by Hill functions in the case that $\Omega = R_m$.

In this § we shall study the case if Ω is a bounded domain and $\Omega^\bullet \in C^\infty$.

First let us state a theorem about approximation on Ω .

<u>Theorem 3.1</u>. Let $f \in W_2^\ell(\Omega)$, $\ell \geq 0$ and let $0 \leq \leq s \leq \ell$, $\Omega^\bullet \in C^\infty$. Then for $k \geq s$ there exists $g \in \mathcal{V}_h^{t,k}(R_m)$ such that

$$\| g - f \|_{W_2^s(\Omega)} \leq C h^{\mu} \| f \|_{W_2^\ell(\Omega)} \tag{3.1}$$

where

$$\mu = \min(\ell - s,\ t - s) \tag{3.2}$$

and C does not depend on h and f .

Proof. The proof follows immediately by the application of the well known continuation theorem. See e.g. [19]. By this theorem there exists a function $F \in W_2^{\ell}(R_m)$ such that $F = f$ on Ω and $\|F\|_{W_2^{\ell}(R_m)} \le C\|f\|_{W_2^{\ell}(\Omega)}$ where C does not depend on f .

Using the basic property of $(t-k)$-regularity there exists $\varphi \in \mathcal{T}_h^{t,k}(R_m)$ such that

$$\|\varphi - F\|_{W_2^{s}(R_m)} \le Ch^{\mu}\|F\|_{W_2^{\ell}(R_m)} \le Ch^{\mu}\|f\|_{W_2^{\ell}(\Omega)} .$$

By restriction to Ω we obtain our theorem.

Remark. Everywhere we assume that $\dot{\Omega} \in C^{\infty}$. Because the continuation theorem is valid for Lipschitz domain theorem 3.1 holds in the general case when Ω is a Lipschitz domain.

The theorem 3.1 solves the problem of the approximation on Ω . Let us prove a theorem dealing with the approximation on the boundary $\dot{\Omega}$.

Theorem 3.2. Let $f \in W_2^{\ell}(\dot{\Omega})$, $\ell > 0$ and let $0 < s \le \ell$. Then for $t \ge s + \frac{1}{2}$, $k \ge s + \frac{1}{2}$ there exists $\varphi \in \mathcal{T}_h^{t,k}(R_m)$ such that

$$\|f - \dot{\varphi}\|_{W_2^{s}(\dot{\Omega})} \le Ch^{\mu}\|f\|_{W_2^{\ell}(\Omega)} \tag{3.3}$$

where

$$\mu = \min(\ell - s, t - s - \tfrac{1}{2}) \tag{3.4}$$

and C does not depend on h and f .

Proof. Using the "inverse" embedding theorem (see e.g. [18] or [20]) there exists $F \in W_2^{\ell+\frac{1}{2}}(R_m)$ such that $F' = f$ and

$$\|F\|_{W_2^{\ell+\frac{1}{2}}(R_m)} \le C\|f\|_{W_2^{\ell}(\Omega')} ,$$

where C does not depend on f . Therefore there exists $g \in \mathcal{M}_h^{t,k}(R_m)$ such that

$$\|F - g\|_{W_2^{s+\frac{1}{2}}(R_m)} \le C h^{\mu} \|F\|_{W_2^{\ell+\frac{1}{2}}(R_m)} \le$$
$$\le C h^{\mu} \|f\|_{W_2^{\ell}(\Omega')}$$

and

$$\mu = \min(\ell - s, t - s - \frac{1}{2}) .$$

Applying the embedding theorem we have

$$\|f - g'\|_{W_2^{s}(\Omega')} = \|F' - g'\|_{W_2^{s}(\Omega')} \le$$
$$\le C\|F - g\|_{W_2^{s+\frac{1}{2}}(R_m)} \le C h^{\mu} \|f\|_{W_2^{\ell}(\Omega')}$$

and the theorem is completely proved.

Remarks. (1) Obviously we do not need $\Omega' \in C^{\infty}$, but we need that Ω' be sufficiently smooth.

(2) Comparing Theorem 3.1 and Theorem 3.2, especially the expressions (3.2) and (3.4), we see that we lost $\frac{1}{2}$ in the second term and we need in Theorem 3.2 larger regularity functions than in Theorem 3.1.

So far it is not clear whether this situation occurs because of the way we prove it or whether this is necessary for the theorem itself. The first term in (3.2) and (3.4) is an optimal one. It follows in general by applying the theory of the n-width or in special case by Theorem 4.2 in [1].

(3) In Theorem 3.2 we did assume that $\ell > 0$ and $s > 0$. In Theorem 3.1 we assumed $\ell \ge 0$, $s \ge 0$. The

assumption $l > 0$, $s > 0$ stems from using the embedding theorem.
This theorem does not hold for $l = 0$ or $s = 0$ (see e.g. Theorem 9.5 [17]). Perhaps the theorem holds for $l = 0$ (resp. $s = 0$).

4. Simultaneous Approximation on Ω and Ω° . Using the results of the § 3 we may approximate the functions $f \in W_2^l(\Omega)$ (resp. $f \in W_2^l(\Omega^{\circ})$) by $g \in \mathcal{V}_h^{t,k}(R_m)$ (resp. g°).

In applications another problem plays an important role. It is the problem of simultaneous approximation on Ω and Ω° .
Let us introduce the space $W_2^{l,m}(\Omega) \subset W_2^l(\Omega)$ with the norm

$$\|f\|^2_{W_2^{l,m}(\Omega)} = \|f\|^2_{W_2^l(\Omega)} + \|f^{\circ}\|^2_{W_2^m(\Omega^{\circ})} .$$

Obviously using the embedding theorem we see that for $m \leq l - \frac{1}{2}$ we have $\|f\|_{W_2^{l,m}(\Omega)} \leq C\|f\|_{W_2^l(\Omega)}$ and hence $W_2^{l,m}(\Omega) = W_2^l(\Omega)$.

If $m > l - \frac{1}{2}$ then the space $W_2^{l,m}(\Omega)$ is smaller than the space $W_2^l(\Omega)$.

The problem of the simultaneous approximation with a weight τ is the problem to find μ such that for every $f \in W_2^{l,m}(\Omega)$ there exists $g \in \mathcal{V}_h^{t,k}(R_m)$ such that

$$\|f-g\|_{W_2^{s_1}(\Omega)} + h^{-\tau}\|f^{\circ}-g^{\circ}\|_{W_2^{s_2}(\Omega^{\circ})} \leq Ch^{\mu}\|f\|_{W_2^{l,m}(\Omega)}$$

and C does not depend on f and h .

The most important case in applications is the case of $\tau > 0$.

As introduction let us prove a theorem which was in fact proved in [13].

<u>Theorem 4.1.</u> Let $f \in W_2^{\ell}(\Omega)$, $\ell \geq 1$. Then for $k \geq 1$, $t \geq 1$ and $\tau \leq t - \frac{1}{2} - \varepsilon$, $\tau \leq \ell - \frac{1}{2} - \varepsilon$, $\varepsilon > 0$ arbitrary, there exists $g \in \mathcal{T}_h^{t,k}(R_m)$ such that

$$(4.1) \qquad \| f - g \|_{W_2^1(\Omega)} + h^{-\tau} \| f' - g' \|_{W_1^0(\Omega')} \leq \leq C h^{\mu} \| f \|_{W_2^{\ell}(\Omega)}$$

where

$$(4.2) \qquad \mu = \min(\ell - 1, t - 1, \ell - \tfrac{1}{2} - \varepsilon - \tau, t - \tfrac{1}{2} - \varepsilon - \tau)$$

and C does not depend on f and h (generally depends on ε).

<u>Proof</u>. Obviously, $f \in W_2^{\ell, \ell - \frac{1}{2}}(\Omega)$. There exists $g \in \mathcal{T}_h^{t,h}(R_m)$ such that (see Theorem 3.1)

$$\| g - f \|_{W_2^{s}(\Omega)} \leq C h^{\mu} \| f \|_{W_2^{\ell}(\Omega)}$$

and

$$\mu = \min(\ell - s, t - s).$$

Using $s = 1$ and $s = \frac{1}{2} + \varepsilon$, $\varepsilon > 0$ and the imbedding theorem we get

$$\| f - g \|_{W_2^1(\Omega)} + h^{-\tau} \| f' - g' \|_{W_2^0(\Omega')} \leq$$
$$\leq C [h^{\mu_1} + h^{-\tau} h^{\mu_2}] \| f \|_{W_2^{\ell}(\Omega)}$$

where

$$\mu_1 = \min(\ell - 1, t - 1),$$

$$\mu_2 = \min(\ell - \frac{1}{2} - \varepsilon, t - \frac{1}{2} - \varepsilon).$$

Theorem 4.1 will not change if instead $f \in W_2^{\ell}(\Omega)$ we have $f \in W_2^{\ell,m}(\Omega)$ with $m > \ell - \frac{1}{2}$.

Intuitively it is possible to see that perhaps the result is too restrictive because of simplicity of the proof. Let us show that we may really get much better results.

Theorem 4.2. Let $f \in \mathring{W}_2^{\ell}(\Omega)$, $\ell \geq s \geq 0$, ℓ, s integral. Then for $k \geq s$, $t \geq s$ there exists $g \in \mathcal{T}_h^{t,k}(R_m)$ such that $g^{\circ} = 0$ and such that

$$\| f - g \|_{W_2^{s}(\Omega)} \leq C h^{\mu} \| f \|_{W_2^{\ell}(\Omega)}$$

where

$$\mu = \min(\ell - s, t - s).$$

Proof. (1) Let $\varphi(x)$, $x \in \Omega$ be the distance of the point x to the boundary Ω°. Because $\Omega^{\circ} \in C^{\infty}$ the function $\varphi(x)$ has all derivatives in the neighborhood of Ω°. Let $\psi(x)$, $x \in R_1$, $0 \leq x$ be a function with all derivatives and such that

$$\psi(x) = 1 \quad \text{for } 0 \leq x \leq 1,$$
$$\psi(x) = 0 \quad \text{for } x \geq 2.$$

Such a function obviously exists. Let further denote $\psi_h(x) = \psi(\frac{x}{h})$ and let $\varphi_h(x) = \psi_h(\varphi(x))$. The function $\varphi_h(x)$ is defined on Ω and $\varphi_h(x) = 1$ on Ω_h°, and $\varphi_h(x) = 0$ on Ω_{2h}.

Further for sufficiently small h the function φ_h has all derivatives and $|D^j \varphi_h| \leq \frac{1}{h^{|j|}} C$.

(2) Let $f \in \mathring{W}_2^{\ell}(\Omega)$. Denote $f_h = f\,\varphi_h(x)$.

Let us show that for $\ell \geq s \geq 0$, $\|f_h\|_{W_2^s(\Omega)} \leq C h^{\ell-s} \|f\|_{W_2^\ell(\Omega)}$.

Because $\Omega' \in C^\infty$, using partition of unity and local transformation of Ω' to a plane, we may restrict ourselves to proving the following special estimate.

Let $f \in C^\infty(R_m)$, $f = 0$ on R_m^- and let f have compact support.

Let $\varphi_h(x) = \psi_h(x)$ and $f_h = f\varphi_h$.

Let us show that for $\ell \geq s \geq 0$

(4.3) $$\|f_h\|_{W_2^s(R_m)} \leq C h^{\ell-s} \|f\|_{W_2^\ell(R_m)} .$$

In fact because $\varphi_h(x) = 0$ for $x_m \geq 2h$ we have $f_h(x) = 0$ for $x_m \geq 2h$.

On the other hand we have

$$\left| \frac{\partial^j \varphi_h(x)}{\partial x_1^{i_1} \dots \partial x_m^{i_m}} \right| \leq \frac{C}{h^{i_m}} .$$

Therefore to prove (4.3) it is sufficient to show that

$$\int_0^{2h} \left[\frac{\partial^j f(x)}{\partial x_1^{i_1} \dots \partial x_m^{i_m}}\right]^2 dx_m = \int_{-\infty}^{2h} \left[\frac{\partial^j f(x)}{\partial x_1^{i_1} \dots \partial x_m^{i_m}}\right]^2 dx_m \leq$$

$$\leq C h^{2(\ell-j)} \int_{-\infty}^{2h} \left[\frac{\partial^\ell f(x)}{\partial x_1^{i_1} \dots \partial x_m^{i_m}}\right]^2 dx_m .$$

Writing $g(x) = \dfrac{\partial^j f}{\partial x_1^{i_1} \dots \partial x^{i_m}}$ we have $\dfrac{\partial^\ell f}{\partial x_1^{i_1} \dots \partial x_m^{i_m+\ell-j}} = \dfrac{\partial^{\ell-j} g}{\partial x_m^{\ell-j}}$.

Obviously we have $g = 0$ identically on R_m^-.

Therefore we may write

$$g(x) = \frac{1}{(\ell-j-1)!} \int_{-\infty}^{x_m} (x_m - t)^{\ell-j-1} \frac{\partial^{\ell-j} g}{\partial x_m^{\ell-j}} (x_1, \dots, x_{m-1}, t)\, dt .$$

Using now the Hardy inequality (see Theorem 3.29 of [21] we get

$$\int_{-\infty}^{2h} q^2(x)\,dx_m = \int_0^{2h} q^2(x)\,dx_m \le C\int_{-\infty}^{2h} (2h - x_m)^{2(\ell-j)}$$
$$\left(\frac{\partial^{\ell-j} q}{\partial x^{\ell-j}}\right)^2 dx_m \le C h^{2(\ell-j)} \int_{-\infty}^{+\infty} \left(\frac{\partial^{\ell-j} q}{\partial x_m^{\ell-j}}\right)^2 dx_m \,.$$

Therefore (4.3) is proved.

(3) In (2) we proved that every function $f \in \overset{\circ}{W}{}_2^{\ell}(\Omega)$ can be written in the form

$$f = f_{H,1} + f_{H,2}$$

where

$$\| f_{H,1} \|_{W_2^{\ell}(\Omega)} \le C \| f \|_{W_2^{\ell}(\Omega)} \quad ,$$

$$\| f_{H,2} \|_{W_2^{\ell}(\Omega)} \le C \| f \|_{W_2^{\ell}(\Omega)}$$

and $f_{H,2} = 0$ on Ω°_H ,

$$\| f_{H,1} \|_{W_2^{s}(\Omega)} \le C H^{\ell-s} \| f \|_{W_2^{\ell}(\Omega)}$$

and $H > 0$ is an arbitrary number.

Choosing $H = \alpha h$ for a proper α and using the basic property of the (t,k)-regularity we may find $q \in \mathcal{T}_h^{t,k}(R_m)$, $q = 0$ on Ω° and such that

$$\| q - f_{H,2} \|_{W_2^{s}(\Omega)} \le C h^{\mu} \| f_{H,2} \|_{W_2^{\ell}(\Omega)}$$

where

$$\mu = \min(\ell - s, t - s) \,.$$

Therefore we get

$$\| q - f \|_{W_2^{s}(\Omega)} \le C h^{\mu} \| f \|_{W_2^{\ell}(\Omega)}$$

and the theorem is proved.

Theorem 4.2 may be easily rewritten for ℓ, s nonintegral.

Theorem 4.3. Let $f \in \overset{\circ}{W}{}_2^{\ell}(\Omega)$, $\ell \geq s \geq 0$. Further let s^* be the smallest integer such that $s^* \geq s$. Then for $k \geq s^*$ and $t \geq s^*$ there exists $g \in \mathcal{M}_h^{t,k}(R_n)$ such that $g^{\cdot} = 0$ and such that

$$\| f - g \|_{W_2^s(\Omega)} \leq C h^{\mu} \| f \|_{W_2^{\ell}(\Omega)}$$

where

$$\mu = \min(\ell - s, t - s) .$$

The proof follows immediately from the well known interpolation theorems.

Let us remark that for ℓ = integral + $\frac{1}{2}$ the space $\overset{\circ}{W}{}_2^{\ell}(\Omega)$ should be understood in a proper way. See § 2.

Theorem 4.4. Let $f \in W_2^{\ell}(\Omega)$, $\Omega^{\cdot} \in C^{\infty}$, ℓ integral and such that $\frac{\partial^j f}{\partial n^j} = 0$ on $\Omega^{\cdot}$ for $j = 0,1,\dots, r \leq \ell - 1$.

Then for $t \geq \ell$ and $s > \tau + \frac{1}{2} + r$, $s \geq \rho \geq 0$, $\sigma \geq 0$, $\tau \geq 0$ 1) there exists $g \in \mathcal{M}_h^{t,s}(R_n)$ such that

$$\| f - g \|_{W_2^{\rho}(\Omega)} + \sum_{j=0}^{r} h^{-\sigma + j} \| \frac{\partial^j g}{\partial n^j} \|_{W_2^{\tau}(\Omega^{\cdot})} \leq$$

$$\leq C(\varepsilon) h^{\mu - \varepsilon} \| f \|_{W_2^{\ell}(\Omega)}$$

where

$$\mu = \frac{t - \chi}{t - \rho} (\ell - \rho); \quad \chi = \max(\rho, \tau + \frac{1}{2} + \sigma)$$

with $\varepsilon > 0$ arbitrary and $C(\varepsilon)$ does not depend on h and f

Proof. Let us define functions g_j, $j = 0, 1, \dots, \ell - 1$ defined on $\Omega^{\cdot}$ such that

----

1) ρ, σ, τ need not be integral.

(4.12) $$q_j = \frac{\partial^j f}{\partial n^j}, \quad j = 0, 1, \dots, l-1 .$$

By assumption we have $q_0 = q_1 = \dots = q_k = 0$.

Let $u \in W_2^l(\Omega)$ be such that

$$\Delta^l u = 0 , \quad \frac{\partial^j u}{\partial n^j} = q_j , \quad j = 0, 1, \dots, l-1 .$$

For the existence of such a function see e.g. [17].

Obviously the function u may be written in a form of a sum namely

(4.13) $$u = \sum_{i=k+1}^{l-1} u_i$$

where

$$\Delta^l u_i = 0$$

and

$$\frac{\partial^j u_i}{\partial n_j} = q_j \vartheta_{i,j} , \quad i = k+1, \dots, l-1 ,$$
$$j = 0, 1, \dots, l-1$$

with

$$\vartheta_{i,j} = 1 \text{ for } j = i ,$$
$$= 0 \text{ for } j \neq i .$$

Because of the embedding theorem we have

(4.14) $$\| q_j \|_{W_2^{l-j-\frac{1}{2}}(\Omega')} \leq C \| f \|_{W_2^l(\Omega)} .$$

Further using the basic theorem about the regularity of the solution (see [17], p.203) we obtain

(4.15) $$\| u_i \|_{W_2^s(\Omega)} \leq C \| q_i \|_{W_2^{s-i-\frac{1}{2}}(\Omega')}$$

for arbitrary s .

Let us remark that $\frac{\partial^j u_i}{\partial n^j} = 0$ on Ω' for $j = 0, 1, \ldots, r$ and $i = r+1, \ldots, \ell - 1$.

In § 2 we introduced the operator Λ_H. Let us denote

$$q_{i,H} = \Lambda_H q_i \tag{4.16}$$

and let $u_{i,H}$ be an analogous function to the function u_i applying $q_{i,H}$ instead q_i. Using Lemma 2.1 and (4.15) for $t \geq s - i - \frac{1}{2}$ we get

$$\| u_i - u_{i,H} \|_{W_2^s(\Omega)} \leq C H^{t-(s-i-\frac{1}{2})} \| q_i \|_{W_2^t(\Omega')} \tag{4.17}$$

and for $t \leq s - i - \frac{1}{2}$ we obtain

$$\| u_{i,H} \|_{W_2^s(\Omega)} \leq C H^{t-(s-i-\frac{1}{2})} \| q_i \|_{W_2^t(\Omega')} . \tag{4.18}$$

Using (4.14) we obtain for $\ell \geq \varphi$

$$\| u_i - u_{i,H} \|_{W_2^\varphi(\Omega)} \leq C H^{\ell - i - \frac{1}{2} - (\varphi - i - \frac{1}{2})} \| f \|_{W_2^\ell(\Omega)} = C H^{\ell - \varphi} \| f \|_{W_2^\ell(\Omega)} . \tag{4.19}$$

For $\varphi \geq \ell$ we obtain

$$\| u_{i,H} \|_{W_2^\varphi(\Omega)} \leq C H^{-(\varphi - \ell)} \| f \|_{W_2^\ell(\Omega)} . \tag{4.20}$$

Hence for $\varphi \leq \ell$

$$\| u - \sum_{i=r+1}^{\ell-1} u_{i,H} \|_{W_2^\varphi(\Omega)} \leq C H^{\ell-\varphi} \| f \|_{W_2^\ell(\Omega)} . \tag{4.21}$$

Denoting $v = u - \sum_{r+1}^{\ell-1} u_{i,H}$, then we have $\frac{\partial^j v}{\partial n^j} = 0$ on Ω for $j = 0, 1, \ldots, r$.

It is easy to see that $f - u \in \overset{\circ}{W}{}_2^\ell(\Omega)$.

We may write

$$(4.22) \qquad f = f - u + v + \sum_{k+1}^{\ell-1} u_{i,H} \ .$$

There exist $g_i \in \mathcal{S}_h^{t,s}(R_m)$, $s \geq \rho$ such that for $x \geq \ell$

$$\| g_i - u_{i,H} \|_{W_2^{\rho}(\Omega)} \leq C H^{-(x-\ell)} h^{\mu_i} \| f \|_{W_2^{\ell}(\Omega)}$$

where

$$(4.23) \qquad \mu_i = \min(x - \rho, t - \rho) \ .$$

Because $\dfrac{\partial^j u_{i,H}}{\partial n_j} = 0$ on Ω° for $j = 0,\dots,k$, we have for $s > \tau + j + \dfrac{1}{2}$, $j = 0,\dots,k$

$$(4.24) \qquad \left\| \frac{\partial^j g_i}{\partial n_j} \right\|_{W_2^{\tau}(\Omega)} \leq C H^{-(x-\ell)} h^{\varkappa} \| f \|_{W_2^{\ell}(\Omega)}$$

where

$$(4.25) \qquad \varkappa = \min\left(x - \tau - \frac{1}{2} - j, t - \tau - \frac{1}{2} - j\right) - \varepsilon \ ,$$

$\varepsilon > 0$ arbitrary.

Let us take now $x = t \geq \ell$ (Remark that we did not fix the value of x.)
Under this assumption we have

$$(4.26) \qquad \left\| \sum_{k+1}^{\ell-1} g_i - \sum_{k+1}^{\ell-1} u_{i,H} \right\|_{W_2^{\rho}(\Omega)} \leq$$
$$\leq C H^{-(t-\ell)} h^{t-\rho} \| f \|_{W_2^{\ell}(\Omega)}$$

and for $s > \tau + \dfrac{1}{2} + j$

$$(4.27) \qquad \left\| \frac{\partial^j}{\partial n^j} \sum_{k+1}^{\ell-1} g_i \right\|_{W_2^{\tau}(\Omega)} \leq$$
$$\leq C H^{-(t-\ell)} h^{t-\tau-\frac{1}{2}-j-\varepsilon} \| f \|_{W_2^{\ell}(\Omega)} , \quad \varepsilon > 0 \ .$$

Using the theorem 4.2 we may find $g_0 \in \mathcal{S}_h^{k,s}(R_m)$ such that

$q_0 = 0$ in Ω'_h and

(4.28) $\| q_0 - (f - u) \|_{W_2^\varphi(\Omega)} \leq$
$$\leq C H^{l-\varphi} \| f \|_{W_2^l(\Omega)}$$

and obviously

(4.29) $$\left\| \frac{\partial^j q_0}{\partial n^j} \right\|_{W_2^\tau(\Omega)} = 0 , \quad j = 0, \dots, k .$$

Further using (4.21) we have

(4.30) $$\| v \|_{W_2^\varphi(\Omega)} \leq C H^{l-\varphi} \| f \|_{W_2^l(\Omega)}$$

and
$$\frac{\partial^j v}{\partial n^j} = 0 , \quad j = 0, \dots, k .$$

Let us take $H = h^\alpha$. Then
$$H^{-(t-l)} h^{t-\varphi} = h^{-\alpha t + \alpha l + t - \varphi} = h^{t(1-\alpha) + \alpha l - \varphi} .$$

Therefore denoting
$$q = \sum_{i=k+1}^{l-1} q_i + q_0$$
we have

(4.31) $$\| q - f \|_{W_2^\varphi(\Omega)} + \sum_{j=0}^{k} h^{-\sigma + j} \left\| \frac{\partial^j q}{\partial n^j} \right\|_{W_2^\tau(\Omega')} \leq$$
$$\leq C H^{\mu - \varepsilon} \| f \|_{W_2^l(\Omega)}$$

where

(4.32) $$\mu = \min [\alpha (l - \varphi), t(1-\alpha) + \alpha l - \varphi ,$$
$$t(1-\alpha) + \alpha l - \tau - \frac{1}{2} - \sigma], \ \varepsilon > 0 \text{ arbitrary.}$$

Denoting

(4.33) $$\chi = \max (\varphi, \tau + \frac{1}{2} + \sigma)$$

we get by optimal choice of α

$$(4.34) \qquad \mu = \frac{t-\chi}{t-\rho}(\ell-\rho) \; .$$

Theorem 4.4 assumes that Ω has a smooth boundary. It is not clear whether the theorem holds for domains which are not sufficiently smooth. Certainly we do not need $\Omega' \in C^{\infty}$ but a sufficient smoothness of the boundary is necessary for the proof.

Different situation occurs in Theorem 4.2 where the assumption about smoothness of Ω' may be much weaker. In Theorem 4.4 we have assumed that $\frac{\partial^{\phi} f}{\partial m^{\phi}} = 0$ on Ω' .

Combining the results of Theorem 4.4 with the results of § 3 we obviously obtain the general case.

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ОБ ОДНОМ УСЛОВИИ ЖЁСТКОСТИ 2-го ПОРЯДКА ДВУСВЯЗНОЙ ПОВЕРХНОСТИ ВРАЩЕНИЯ

Н.Г. ПЕРЛОВА, Ростов-на-Дону

Рассмотрим двусвязную поверхность вращения

$S: \bar{r} = u\bar{e} + r(u)\bar{a}(v)$, где $r = r(u) \in C^{(2)}$,

$\frac{1}{r'(u)} \neq 0$, ограниченную параллелями $u = 0$ и $u = a$.

Справедлива

<u>Теорема</u>. Поверхность S обладает жесткостью 2-го порядка относительно бесконечно малых изгибаний, сохраняющих нормальную кривизну одной из ее граничных параллелей.

<u>Доказательство</u>. Бесконечно малое изгибание 1-го порядка поверхности S , не подчинённое граничным условиям, определяется векторным полем [1]

$$\underset{(1)}{\bar{z}}(u,v) = \underset{(1)}{\alpha}(u,v)\bar{e} + \underset{(1)}{\beta}(u,v)\bar{a}(v) + \underset{(1)}{\gamma}(u,v)\bar{a}'(v)$$

класса $C^{(2)}$, удовлетворяющим системе уравнений

$$(1) \quad \begin{cases} \underset{(1)}{\alpha}_u + r'\underset{(1)}{\beta}_u = 0 , \\ \underset{(1)}{\beta} + \underset{(1)}{\gamma}_v = 0 , \\ \underset{(1)}{\alpha}_v + r'(\underset{(1)}{\beta}_v - \underset{(1)}{\gamma}) + r\underset{(1)}{\gamma}_u = 0 . \end{cases}$$

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Бесконечно малое изгибание 2-го порядка поверхности S , не подчинённое граничным условиям, определяется векторным полем [1]

$$\underset{(2)}{\bar{z}}(u,v) = \underset{(2)}{\alpha}(u,v)\,\bar{e} + \underset{(2)}{\beta}(u,v)\,\bar{a}(v) + \underset{(2)}{\gamma}(u,v)\,\bar{a}'(v)$$

класса $C^{(2)}$, удовлетворяющим системе уравнений

$$(2)\quad \begin{cases} \underset{(2)}{\alpha}_u + \varkappa' \underset{(2)}{\beta}_u = -\dfrac{1}{2}\left[\underset{(1)}{\alpha}{}^2_u + \underset{(1)}{\beta}{}^2_u + \underset{(1)}{\gamma}{}^2_u\right], \\ \underset{(2)}{\beta} + \underset{(2)}{\gamma}_v = -\dfrac{1}{2\varkappa}\left[\underset{(1)}{\alpha}{}^2_v + (\underset{(1)}{\beta}_v - \underset{(1)}{\gamma})^2\right], \\ \underset{(2)}{\alpha}_v + \varkappa'(\underset{(2)}{\beta}_v - \underset{(2)}{\gamma}) + \varkappa \underset{(2)}{\gamma}_u = \\ \qquad = -\left[\underset{(1)}{\alpha}_u \underset{(1)}{\alpha}_v + \underset{(1)}{\beta}_u (\underset{(1)}{\beta}_v - \underset{(1)}{\gamma})\right]. \end{cases}$$

Первая и вторая вариации нормальной кривизны линии $u = const$ на поверхности определяются формулам [2]

$$\delta k_n = \frac{\delta N}{G}, \quad \delta^2 k_n = \frac{\delta^2 N}{G},$$

где G и N суть коэффициенты первой и второй основных форм поверхности.

Найдем выражение коэффициента N^* второй основной формы поверхности $\bar{\varkappa}^* = \bar{\varkappa} + \varepsilon \underset{(1)}{\bar{z}} + \varepsilon^2 \underset{(2)}{\bar{z}}$:

$$(3)\qquad N^* = \frac{1}{\sqrt{EG - F^2}} \times$$

$$\times \begin{vmatrix} 1 + \varepsilon \underset{(1)}{\alpha}_u + \varepsilon^2 \underset{(2)}{\alpha}_u & \varkappa' + \varepsilon \underset{(1)}{\beta}_u + \varepsilon^2 \underset{(2)}{\beta}_u & \varepsilon \underset{(1)}{\gamma}_u + \varepsilon^2 \underset{(2)}{\gamma}_u \\ \varepsilon \underset{(1)}{\alpha}_v + \varepsilon^2 \underset{(2)}{\alpha}_v & \varepsilon(\underset{(1)}{\beta}_v - \underset{(1)}{\gamma}) + \varepsilon^2(\underset{(2)}{\beta}_v - \underset{(2)}{\gamma}) & \varkappa + \varepsilon^2(\underset{(2)}{\gamma}_v + \underset{(2)}{\beta}) \\ \varepsilon \underset{(1)}{\alpha}_{vv} + \varepsilon^2 \underset{(2)}{\alpha}_{vv} & -\varkappa + \varepsilon(\underset{(1)}{\beta}_{vv} - \underset{(1)}{\gamma}_v) + \varepsilon^2(\underset{(2)}{\beta}_{vv} - 2\underset{(2)}{\gamma}_v - \underset{(2)}{\beta}) & \varepsilon(\underset{(1)}{\beta}_v - \underset{(1)}{\gamma}) + \varepsilon^2(\underset{(2)}{\gamma}_{vv} + 2\underset{(2)}{\beta}_v - \underset{(2)}{\gamma}) \end{vmatrix}$$

откуда

$$\delta N = \frac{1}{\kappa\sqrt{1+\kappa'^2}}\left[\kappa\,(\kappa'\underset{(1)}{\alpha}_{vv} + \kappa\,\underset{(1)}{\alpha}_{u}) - \kappa\,(\underset{(1)}{\beta}_{vv} - \underset{(1)}{\gamma}_{v})\right].$$

Из (1) следует:

$$\underset{(1)}{\alpha}_{vv} = -\kappa'(\underset{(1)}{\beta}_{vv} - \underset{(1)}{\gamma}_{v}) - \kappa\,\underset{(1)}{\gamma}_{uv} ,$$

$$\underset{(1)}{\alpha}_{u} = -\kappa'\,\underset{(1)}{\beta}_{u} = \kappa'\,\underset{(1)}{\gamma}_{uv} .$$

Поэтому

$$\delta N = -\sqrt{1+\kappa'^2}\,(\underset{(1)}{\beta}_{vv} - \underset{(1)}{\gamma}_{v}) = -\sqrt{1+\kappa'^2}\,(\underset{(1)}{\beta}_{vv} + \underset{(1)}{\beta}) .$$

Пользуясь периодичностью функций $\underset{(1)}{\alpha}$, $\underset{(1)}{\beta}$, $\underset{(1)}{\gamma}$, представим их в виде рядов Фурье:

$$(4)\begin{cases} \underset{(1)}{\alpha}(u,v) = \sum\limits_{k=0}^{+\infty}[\underset{(1)}{\varphi}_k(u)e^{ikv} + \underset{(1)}{\bar\varphi}_k(u)e^{-ikv}] , \\ \underset{(1)}{\beta}(u,v) = \sum\limits_{k=0}^{+\infty}[\underset{(1)}{\psi}_k(u)e^{ikv} + \underset{(1)}{\bar\psi}_k(u)e^{-ikv}] , \\ \underset{(1)}{\gamma}(u,v) = \sum\limits_{k=0}^{+\infty}[\underset{(1)}{\chi}_k(u)e^{ikv} + \underset{(1)}{\bar\chi}_k(u)e^{-ikv}] , \end{cases}$$

где функции $\underset{(1)}{\varphi}_k$, $\underset{(1)}{\psi}_k$, $\underset{(1)}{\chi}_k$ удовлетворяют системе уравнений [1]

$$(5)\begin{cases} \underset{(1)}{\varphi}'_k + \kappa'\underset{(1)}{\psi}'_k = 0 , \\ ik\,\underset{(1)}{\chi}_k + \underset{(1)}{\psi}_k = 0 , \\ ik\,\underset{(1)}{\varphi}_k + \kappa'(ik\,\underset{(1)}{\psi}_k - \underset{(1)}{\chi}_k) + \kappa\,\underset{(1)}{\chi}'_k = 0 \end{cases}$$

при натуральном $k \geq 2$; исключение неизвестных функций $\underset{(1)}{\varphi}_k$ и $\underset{(1)}{\psi}_k$ из (4) приводит к уравнению

(6) $$\kappa\,\underset{(1)}{\chi}''_k + (k^2-1)\kappa''\underset{(1)}{\chi}_k = 0 \qquad (k \geq 2)$$

для одной неизвестной функции $\underset{(1)}{\chi}_k$, с помощью которой $\underset{(1)}{\varphi}_k$ и $\underset{(1)}{\psi}_k$ находятся без квадратур.

В силу (4_2) условие $\delta k_m/_{u=const} = 0$ принимает вид

$$\sum_{k=0}^{+\infty} (k^2-1)[\underset{(1)}{\psi}_k(u)e^{ikv} + \underset{(1)}{\overline{\psi}}_k(u)e^{-ikv}]/_{u=const} = 0,$$

откуда в силу линейной независимости функций e^{ikv} получаем:

$$\underset{(1)}{\psi}_k(u)/_{u=const} = 0 \qquad (k \geq 2)$$

(при $k=0$ $\underset{(1)}{\psi}_k(u) \equiv 0$ [1]) или, что то же,

$$\underset{(1)}{\chi}_k(u)/_{u=const} = 0 \qquad (k \geq 2) \ .$$

Предположим, что поверхность S допускает бесконечно малое изгибание 1-го порядка с сохранением нормальной кривизны обеих граничных параллелей. Тогда существует решение уравнения (6) при одном или нескольких $k \geq 2$ удовлетворяющее условиям

(7) $$\underset{(1)}{\chi}_k(0) = \underset{(1)}{\chi}_k(a) = 0 \ .$$

Заметим, что поверхность S в этом случае невыпуклая: если $\varkappa'' < 0$, то решение уравнения

$$\underset{(1)}{\chi}''_k + (k^2-1)\frac{\varkappa''}{\varkappa}\underset{(1)}{\chi}_k = 0$$

не может иметь более одного нуля в интервале $[0, a]$. Если же $\varkappa'' > 0$, то есть поверхность S строго невыпуклая, то на ней существует счетное множество параллелей $u = a_k$ таких, что выполняются условия

$$\underset{(1)}{\chi}_k(0) = \underset{(1)}{\chi}_k(a_k) = 0 .$$

Действительно, при $\varkappa'' > 0$ расстояние между двумя последовательными нулями решения уравнения (6) меньше $\dfrac{\pi}{\sqrt{\min (k^2-1)\frac{\varkappa''}{\varkappa}}}$ [3] и, следовательно, при всех k, больших некоторого k_0, оно меньше a.

Рассмотрим продолжение 2-го порядка бесконечно малого изгибания $\underset{(1)}{\bar{x}} = \sum\limits_k \underset{(1)}{x}_k + \underset{(1)}{\bar{x}}_1 + \underset{(1)}{\bar{x}}_0$, где суммирование ведется по тем значениям $k \geq 2$, для которых выполняется условие (7_1).

Из (3), учитывая (7_1), найдем:

$$\begin{aligned}\delta^2 N|_{u=0} = \frac{1}{\varkappa\sqrt{1+\varkappa'^2}}\Big[&\varkappa(\varkappa' \underset{(2)}{\alpha}_{vv} + \varkappa \underset{(2)}{\alpha}_u) - \varkappa(\underset{(2)}{\beta}_{vv} - \underset{(2)}{\gamma}_v) +\\ &+ 2\varkappa(\underset{(2)}{\gamma}_v + \underset{(2)}{\beta}) + \varkappa \underset{(1)}{\alpha}_{vv} \underset{(1)}{\beta}_u - \varkappa \underset{(1)}{\alpha}_v \underset{(1)}{\gamma}_u +\\ &+ \varkappa' \underset{(1)}{\alpha}_v \underset{(1)}{\gamma}_0 + \underset{(1)}{\gamma}_0^2\Big]\Big|_{u=0} .\end{aligned}$$

Так как из (2) следует:

$$\begin{aligned}\underset{(2)}{\alpha}_{vv} = &-\varkappa'(\underset{(2)}{\beta}_{vv} - \underset{(2)}{\gamma}_v) - \varkappa \underset{(2)}{\gamma}_{uv} - [\underset{(1)}{\alpha}_{uv} \underset{(1)}{\alpha}_v + \underset{(1)}{\alpha}_u \underset{(1)}{\alpha}_{vv} +\\ &+ \underset{(1)}{\beta}_{uv}(\underset{(1)}{\beta}_v - \underset{(1)}{\gamma}) + \underset{(1)}{\beta}_u(\underset{(1)}{\beta}_{vv} - \underset{(1)}{\gamma}_v)] ,\end{aligned}$$

$$\begin{aligned}\underset{(2)}{\alpha}_u = &\varkappa' \underset{(2)}{\gamma}_{uv} - \frac{1}{2}[\underset{(1)}{\alpha}^2_u + \underset{(1)}{\beta}^2_u + \underset{(1)}{\gamma}^2_u] + \frac{\varkappa'}{2\varkappa^2}\{\varkappa[2 \underset{(1)}{\alpha}_v \underset{(1)}{\alpha}_{uv} +\\ &+ 2(\underset{(1)}{\beta}_v - \underset{(1)}{\gamma})(\underset{(1)}{\beta}_{uv} - \underset{(1)}{\gamma}_u)] - \varkappa'[\underset{(1)}{\alpha}^2_v + (\underset{(1)}{\beta}_v - \underset{(1)}{\gamma})^2]\} ,\end{aligned}$$

$$2(\underset{(2)}{\beta} + \underset{(2)}{\gamma}_v) = -\frac{1}{\varkappa}[\underset{(1)}{\alpha}^2_v + (\underset{(1)}{\beta}_v - \underset{(1)}{\gamma})^2] ,$$

то

$$\delta^2 N_{/u=0} = -\sqrt{1+\varkappa'^2}\left\{\beta_{(2)vv} + \beta_{(2)} - \frac{\varkappa}{2}\left[\sum_m \gamma_{(1)mu v} + \gamma_{(1)1uv}\right]^2 + \right.$$
$$\left. + \varkappa\left[\sum_m \gamma_{(1)mu} + \gamma_{(1)1u}\right]^2 + \frac{1}{2}C_0^2\varkappa\right\}_{/u=0} ,$$

где $C_0\varkappa = \gamma_{(1)0}$ [1].

Предположим, что $\delta^2 \varkappa_{m/u=0} = 0$. Тогда

$$\delta^2 N_{/u=0} = 0 . \tag{8}$$

В силу линейной независимости функций e^{ilv} должны равняться нулю коэффициенты при них в левой части равенства (8), в том числе коэффициент при e^0:

$$-\sqrt{1+\varkappa'^2}\left\{-\frac{1}{\varkappa}\left[\sum_m |\varphi_{(1)m}|^2 m^2 + |\varphi_{(1)1}|^2 + \frac{1}{2}C_0^2\varkappa^2\right] - \right.$$
$$-\varkappa\left[\sum_m |\chi'_{(1)m}|^2 m^2 + |\chi'_{(1)1}|^2\right] + 2\varkappa\left[\sum_m |\chi'_{(1)m}|^2 + |\chi'_{(1)1}|^2\right] +$$
$$\left. + \frac{1}{2}C_0^2\varkappa\right\} = \varkappa\sqrt{1+\varkappa'^2}\sum_m (m^2-1)|\chi'_{(1)m}|^2 = 0 .$$

Следовательно, в предположении нежесткости 2-го порядка поверхности S при условии сохранения нормальной кривизны одной только граничной параллели $u = 0$ получаем:

$$\chi'_{(1)m}(0) = 0 \qquad (m \geq 2) .$$

Так как уравнение

$$\chi''_{(1)m} + (m^2-1)\frac{\varkappa''}{\varkappa}\chi_{(1)m} = 0$$

при условиях $\chi_{(1)m}(0) = 0$, $\chi'_{(1)m}(0) = 0$ имеет только нулевое решение в интервале $[0, a]$, то

$$\chi_{(1)m}(u) \equiv 0 \qquad (m \geq 2) ,$$

а потому в силу (5)

$$\underset{(1)}{\varphi}_k(u) \equiv 0\,, \quad \underset{(1)}{\psi}_k(u) \equiv 0 \qquad (k \geq 2)\,.$$

Это означает, что поле $\underset{(1)}{z}(u,v)$ тривиально на всей поверхности S, что противоречит предложению о нежесткости 2-го порядка. Теорема доказана.

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