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UNCONDITIONALLY CONVERGING OPERATORS IN LOCALLY CONVEX HAUSDORFF SPACES

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Abstract: An unconditionally converging operator takes weakly unconditionally convergent series into unconditionally convergent series. These operators form a closed two-sided ideal in L(E,E), the space of all continuous operators from a locally convex Hausdorff space E to E , endowed with the uniform topology on bounded sets.

1. Preliminaries.

All linear operators are to be continuous. (E, τ) and (F, τ') will denote locally convex Hausdorff spaces with topologies τ and τ' respectively.

Definition 1.1. A series $\sum_{i=1}^{\infty} x_i$ in E with topology τ is unconditionally convergent (uc) if it satisfies the following condition:

(A) Subseries convergence: Corresponding to each subseries $\sum_{i=1}^{\infty} x_{k_i}$, there is an element x in E such that $\lim_n \sum_{i=1}^{n} x_{k_i} = x$, the convergence being relative to τ .

In [3] the following conditions are proven equivalent to (A).

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Let E' denote the space of τ -continuous linear functionals on E . Let $w(E,E')$ be the weakest topology on E for which all the maps in E' are continuous. ($w(E,E')$ is called the weak topology on E .) Then

(B) $\sum_{i=1}^{\infty} x_i$ is subseries convergent relative to the $w(E,E')$ topology for E .

Let $S = \{\sum_{i\in\sigma}^{\infty} x_i : \sigma$ finite$\}$. Then

(C) S is precompact (totally bounded) relative to τ .

(D) The $w(E,E')$ closure of S is $w(E,E')$ compact.

<u>Definition 1.2</u>. A series $\sum_{i=1}^{\infty} x_i$ of elements of (E,τ) is said to be wekly unconditionally convergent (wuc) if $\sum_{i=1}^{\infty} |f(x_i)| < \infty$ for every f in E' .

<u>Remark 1</u>: If $\sum_{i=1}^{\infty} x_i$ is a wuc series, then $f(S)= \{\sum_{i\in\sigma} f(x_i): \sigma$ finite$\}$ is bounded for every f in E'. Therefore by Theorem 3, p.409 of [2], $S = \{\sum_{i\in\sigma} x_i : \sigma$ finite$\}$ is bounded.

<u>Definition 1.3</u>. A linear operator $T: E \to F$ is said to be unconditionally converging (uc operator) if it sends every wuc series in E into uc series in F .

<u>Definition 1.4</u>. A linear operator $T: E \to F$ is said to be boundedly weakly compact if the $w(F,F')$ closure of $T(S)$ is $w(F,F')$ compact where S is any τ bounded subset of E .

<u>Remark 2</u>: In normed linear spaces this definition of a boundedly weakly compact operator is equivalent to: $T: X \to Y$ is weakly compact operator if the weak clo-

sure of $T(S)$ is compact in the weak topology of Y where S is the unit sphere in X.

We now give a result for locally convex spaces which is a consequence of Orlicz's Theorem for Banach spaces.

Proposition 1.5. Let (E, τ) and (F, τ') be locally convex Hausdorff spaces and $T: E \to F$. Then if T is a boundedly weakly compact operator, T is a uc operator.

Proof: Let $\sum_{i=1}^{\infty} x_i$ be a wuc series and $S = \{\sum_{i\in\sigma} x_i : \sigma \text{ finite}\}$. By Remark 1 of this section, S is τ bounded. Since T is a boundedly weakly compact operator, the $w(F, F')$ closure of $T(S) = \{\sum_{i\in\sigma} Tx_i : \sigma \text{ finite}\}$ is $w(F, F')$ compact. Hence by 1.1-(D), $\sum_{i=1}^{\infty} Tx_i$ is a uc series.
Therefore T is a uc operator.

2. The space $L(E,F)$.

We now consider $L(E,F)$, the space of all continuous operators from E to F, endowed with the uniform topology on bounded sets. An σ-neighborhood base for the uniform topology on bounded sets for $L(E,F)$ consists of all sets $M(S,H) = \{f \in L(E,F) : f(S) \subseteq H\}$ where S is a bounded subset of E and H belongs to the σ-neighborhood base of F.

In the case of normed linear spaces, the uniform topology on bounded sets is the uniform operator topology.

Proposition 2.6. Let $UC(E,F)$ denote all uc operators from E to F where E and F are locally convex Hausdorff spaces. Then $UC(E, F)$ is closed in $L(E, F)$ where $L(E, F)$ has the uniform topology on bounded sets.

Proof: Let T_n, $n \in I$, be a net of uc operators and $\{T_n\} \to T$. Let $\sum_{i=1}^{\infty} x_i$ be an arbitrary wuc series in E. Then
$S = \{\sum_{i \in \sigma} x_i : \sigma$ finite$\}$ is bounded and $T(S) =$
$= \{\sum_{i \in \sigma} T_n x_i : \sigma$ finite$\}$ is precompact for every $n \in I$ by 1.1-(C).

Let K be an arbitrary σ-neighborhood in F. There exists an σ-neighborhood H in F such that $H + H \subseteq$
$\subseteq K$. Since S is bounded, $M(S, H)$ is an open σ-neighborhood in the σ-neighborhood base of $L(E, F)$. Now $\{T_n\} \to T$ implies there exists $k \in I$ such that $T -$
$- T_n \in M(S, H)$ for all $n \geq k$. Since H is an σ-neighborhood, there exists a finite set B in F such that $T_k(S) = \{\sum_{i \in \sigma} T_k(x_i) : \sigma$ finite$\} \subseteq B + H$.

Since $T(S) \subseteq T(S) - T_k(S) + T_k(S) \subseteq T(S) - T_k(S) + B + H$
$\subseteq H + B + H \subseteq B + K$, $T(S) = \{\sum_{i \in \sigma} T(x_i) : \sigma$ finite$\}$ is precompact and hence by 1.1-(C), $\sum_{i=1}^{\infty} T(x_i)$ is a uc series. So T is a uc operator.

Proposition 2.7. Linear combinations of uc operators are uc. The product of a uc operator and a linear operator is uc.

Proof: Let T and S be uc operators from E to F, and let $\sum_n x_n$ be an arbitrary wuc series in E. Since T and S are uc operators, $\sum_n T x_n$ and $\sum_n S x_n$ are uc series. Hence $\Sigma_n (Tx_n + Sx_n) = \Sigma_n (T+S)(x_n)$ is a uc series and therefore $T + S$ is a uc operator. Clearly αT is a uc operator. So linear combinations of uc operators are uc.

Since continuous maps preserve wuc and uc series, the product of a uc operator and a bounded linear operator is uc.

Theorem 2.8. Let $L(E, E)$ have the uniform topology on bounded sets. Then the uc operators form a closed two-sided ideal in $L(E, E)$.

Proof : This follows from Propositions 2.6 and 2.7.

R e f e r e n c e s

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[3] C.W. McARTHUR: On a Theorem of Orlicz and Pettis, Pacific J.Math.22(1967),297-302.

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