Commentationes Mathematicae Universitatis Carolinae

13,4 (1972)

# FIXED POINT THEOREMS FOR GENERALIZED CONTRACTIONS

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W.V. Petryshyn has given in [7] some fixed point theorems on so called [3],[4] "generalized contractions" (Def. 1 (i)) and on "uniformly generalized contractions" (Def. 1 (ii)) proving them by a degree argument (and therefore function's domains must have interior points). We strengthen and generalize some of these results by a unifying and elementary approach, using methods discussed in [3],[4],[5],[8],[9].

<u>Definition 1</u>: Let $(E, \| \ \|)$ be a normed linear space and $\emptyset \neq X \subset E$ ;

(i) $f: X \longrightarrow E$ is said to be a "generalized contraction": $\Longleftrightarrow$

$$(*) \bigvee_{\alpha: X \to [0,1)} \bigwedge_{x,y \in E} (x,y) \in X \times X \Longrightarrow \|f(x) - f(y)\| \leq \leq \alpha(x) \|x - y\| ,$$

(ii) $f: E \longrightarrow E$ is said to be a "uniformly generalized contraction with respect to $X$ ": $\Longleftrightarrow$

$$(**) \bigvee_{\alpha: E \to [0,1)} \bigwedge_{x,y \in E} (x,y) \in E \times X \Longrightarrow \|f(x) - f(y)\| \leq \leq \alpha(x) \|x - y\| .$$

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AMS, Primary: 47H10 Ref. Ž. 7.978.53

<u>Remark 1:</u>

1) Contractions in the sense of Banach are generalized contractions.

2) [4]: Let $(E, \| \|)$ be a normed linear space and suppose $\emptyset \neq X \subset E$ is open, bounded and convex; let $f : X \longrightarrow E$ be continuously (Fréchet) differentiable. Then $f$ is a generalized contraction iff $\| f'_x \| < 1$ for all $x \in X$. A similar example may be given satisfying condition $(**)$, see [3].

<u>Theorem 1</u>: Let $(E, \| \|)$ be a normed linear space and suppose $\mathcal{T}$ is a Hausdorff topology for $E$, such that

(i) $(E, \mathcal{T})$ is a topological linear space,

(ii) $\bigwedge_{S \subset E}$ $S$ convex $\wedge$ $S$ $\mathcal{T}$-compact $\Longrightarrow$ $S$ is norm-bounded,

(iii) $\bigwedge_{x \in E} \bigwedge_{r \geq 0} B(x, r) := \{ y \mid y \in E \wedge \|x - y\| \leq r \} \Rightarrow B(x, r)$ is $\mathcal{T}$-closed.

Let $\emptyset \neq X \subset E$ be a convex $\mathcal{T}$-compact subset of $E$ and suppose $f : X \rightarrow X$ is a generalized contraction.

<u>Then:</u> (a) There is a unique $x_0 \in X$ such that $f(x_0) = x_0$;

(b) For $x \in X$ we have $\lim_{n \rightarrow \infty} \{ f^n(x) \} = x_0$ (strongly).

<u>Proof</u>: (a): Let $\mathcal{V} := \{ S \mid \emptyset \neq S \subset X$, $S$ convex, $\mathcal{T}$-closed and $f(S) \subset S \}$.

We have $\mathcal{T} \neq \emptyset$ $(X \in \mathcal{T})$. Ordered by $S_1 \leq S_2 :\Longleftrightarrow$ $\Longleftrightarrow S_1 \supset S_2$, it can easily be seen, $(\mathcal{T}, \leq)$ being inductively ordered. Let $S_0 \in \mathcal{T}$ be maximal (Zorn). Defining $\delta := diam\,(S_0)$ we have $0 \leq \delta < \infty$ (ii). Assume $\delta > 0$ and let $x \in S_0$; we define $\delta_1 :=$ $:= \alpha(x)\delta$ and $S_1 := S_0 \cap B(f(x), \delta_1)$. We have $\emptyset \neq S_1 \subset X$ $(S_0 \subset X \wedge f(x) \in S_1)$ and $S_1$ is $\mathcal{Z}$-closed by (iii). Finally, we have for $z \in S_1$ $f(z) \in S_0$ and $\| f(x) - f(z) \| \leq \alpha(x) \| x - z \| \leq \alpha(x)\delta \leq \delta_1$, i.e. $f(S_1) \subset S_1$: $S_1 = S_0$ (maximality of $S_0$). This implies $S_0 \subset B(f(x), \delta_1)$. Now define $S_2 :=$ $:= \bigcap_{y \in S_0} S_0 \cap B(y, \delta_1)$. Then $\emptyset \neq S_2 \subset X$ $(S_0 \subset X \wedge f(x) \in S_2)$, $S_2$ is convex and $\mathcal{Z}$-closed by (iii). It is easily verified that $(*)\,\overline{co\,[f(S_0)]}^{\mathcal{Z}} =$ $= S_0$ ($\mathcal{Z}$-closed convex hull) [Take $S_3 := \overline{co\,[f(S_0)]}^{\mathcal{Z}}$ and prove $S_3 \in \mathcal{T}$ and $S_3 \subset S_0$]. Now let $u \in S_2$ and $y \in S_0$.

Then $\| f(u) - f(y) \| \leq \| u - y \| \leq \delta_1$, i.e. $f(S_0) \subset$ $\subset B(f(u), \delta_1)$. It follows $S_0 = \overline{co\,[f(S_0)]}^{\mathcal{Z}} \subset$ $\subset \overline{B(f(u), \delta_1)}^{\mathcal{Z}} \subset B(f(u), \delta_1)$ by (iii), i.e. $f(u) \in \bigcap_{y \in S_0} B(y, \delta_1) \cap S_0$, i.e. $f(u) \in S_2$. The maximality of $S_0$ gives $S_2 = S_0$. Finally let $u, v \in$ $\in S_2$; we have $u \in B(v, \delta_1)$ $(v \in S_0)$ implying $\| u - v \| \leq \delta_1$ and $diam\,(S_2) \leq \delta_1 < \delta = diam\,(S_2)$, a contradiction: We have $\delta = 0$, i.e. there exists $x_0 \in$ $\in X$ such that $S_0 = \{x_0\}$. Because of $f(S_0) \subset S_0$ we have $f(x_0) = x_0$; (b) Let $z \in X$ and $n \in \mathbb{N}$. Then $\| f^n(z) - x_0 \| \leq \| f^n(z) - f(x_0) \| \leq \alpha(x_0) \| f^{n-1}(z) - x_0 \|$

implying (by induction) $\|f^n(x)-x_0\| \le [\alpha(x_0)]^n \|x-x_0\|$ such that $\lim_{n\to\infty}\{f^n(x)\}=x_0$ $(0\le\alpha(x_0)<1)$ ; (b) is proved. The uniqueness of $x_0$ is an immediate consequence of (b) or, directly of $f$'s contraction property $(\|f(x)-f(y)\| \le \|x-y\|$ for $x \neq y$ .

Corollary 1: Let $(E, \| \ \|)$ be a normed linear space, let $\mathcal{T}$ be a Hausdorff topology for $E$ with (i) - (iii) of Theorem 1. Let $R \ge 0$ and suppose $B(0,R)$ is $\mathcal{T}$-compact and $f: B(0,R) \longrightarrow E$ is a generalized contraction such that $\|f(x)\| \le R$ if $\|x\| = R$ (i.e. $f(\mathrm{bd}(B(0,R))) \subset B(0,R))$ .

Then: (a) There exists a unique $x_0 \in B(0,R)$ such that $f(x_0)=x_0$ ;

(b) For $x \in B(0,R)$ we have $\lim_{n\to\infty}\{[\frac{1}{2}(Id+f)]^n(x)\}=x_0$ (strongly).

Proof (see [4]): Define $g: B(0,R) \longrightarrow E$ by $g := \frac{1}{2}(Id+f)$. Then we have $g(B(0,R)) \subset B(0,R)$, $g$ is a generalized contraction, the fixed point sets of $f$ and $g$ are the same. Theorem 1 completes the proof.

Remark 2:

Examples for $\mathcal{T}$ :

1) Let $(E, \| \ \|)$ be a conjugate space and let $\mathcal{T}$ be the weak* topology for $E$ . Then (i) - (iii) of Theorem 1 comes true.

2) Let $(E, \| \ \|)$ be a reflexive Banach space and let $\mathcal{T}$ be the weak topology for $E$ . Then (i) - (iii) of Theorem 1 comes true.

3) W.A. Kirk [4] proves Theorem 1 and Corollary 1 in the

case of a conjugate space $(E, \| \, \|)$ and the weak* topology for $E$ .

Theorem 2: Let $(E, \| \, \|)$ be a normed linear space, suppose $\mathcal{T}$ is a Hausdorff topology for $E$ , such that

(i) $(E, \mathcal{T})$ is a topological linear space,

(ii) $\bigwedge_{S \subset E}$ $S$ convex $\wedge$ $S$ $\mathcal{T}$-compact $\Rightarrow$ $S$ is norm-bounded,

(iii) $\bigwedge_{x \in E} \bigwedge_{r \geq 0} B(x, r) := \{ y \mid y \in E \wedge \| x - y \| \leq r \} \Rightarrow B(x, r)$ is $\mathcal{T}$-closed,

(iv) The norm topology for $E$ is finer than $\mathcal{T}$ .

Let $\emptyset \neq X \subset E$ be a convex $\mathcal{T}$-compact and $\mathcal{T}$- (sequentially compact) subset of $E$ , let $f : X \longrightarrow E$ be a generalized contraction and $g : [X, \mathcal{T}] \longrightarrow [E, \| \, \|]$ sequentially continuous such that

$$(K_1) \quad \bigwedge_{x, y \in E} (x, y) \in X \times X \Rightarrow f(x) + g(y) \in X \; .$$

Then $f + g$ has a fixed point.

Proof: Let $y \in X$ . We define $h_y : X \longrightarrow X$ $(K_1)$ by $h_y(x) := f(x) + g(y)$ ; $h_y$ is a generalized contraction. By Theorem 1 there is a unique $x_y \in X$ such that $h_y(x_y) = x_y$ . Defining $T : X \longrightarrow X$ by $T(y) := x_y$ we have for $y, z \in X$

$$\| T(y) - T(z) \| \leq \| x_y - x_z \| \leq \| h_y(x_y) - h_z(x_z) \| \leq$$
$$\leq \| f(x_y) - f(x_z) + g(y) - g(z) \| \leq \| f(x_y) - f(x_z) \| +$$
$$+ \| g(y) - g(z) \| \leq \alpha(x_y) \| x_y - x_z \| + \| g(y) - g(z) \| \leq$$
$$\leq \alpha(x_y) \| T(y) - T(z) \| + \| g(y) - g(z) \| \quad ,$$

such that

$$(*) \quad \|T(y) - T(x)\| \leq \frac{1}{1-\alpha(x_y)} \|g(y) - g(x)\| .$$

$T$ is continuous in the norm topology: let $\{x_n\} \in X^{\mathbb{N}}$ and $x_0 \in X$ such that $x_n \to x_0$ (strongly). Then by (iv) $\mathcal{T}-\lim_{n\to\infty} \{x_n\} = x_0$ . Now $g(x_n) \to g(x_0)$ and $\{T(x_n)\} \to T(x_0)$ (strongly) by $(*)$. Let $\{T(x_n)\} \in X^{\mathbb{N}}$, $\{x_n\} \in X^{\mathbb{N}}$ . There is a subsequence $\{x_n'\} \in X^{\mathbb{N}}$ of $\{x_n\} \in X^{\mathbb{N}}$ and $x_1 \in X$ such that $\mathcal{T}-\lim_{n\to\infty} \{x_n'\} = x_1$ ( $X$ is $\mathcal{T}$ - (sequentially compact)). Then $g(x_n') \to g(x_1)$ (strongly), consequently by $(*)$ $\|T(x_n') - T(x_1)\| \leq \frac{1}{1-\alpha(x_{x_1})} \|g(x_n') - g(x_1)\| \to 0$ ,i.e. $\{T(x_n)\}$ has a (strongly) convergent subsequence. Finally $X$ is norm-bounded (ii) and norm-closed, because $X$ is $\mathcal{T}$-closed and $\mathcal{T}$ is coarser than the norm topology. Schauder's fixed point theorem completes the proof (for let $y \in X$ such that $y = T(y)$ then $y = T(y) = x_y$ and $x_y = h(x_y) = f(x_y) + g(y)$ ,i.e. $y = f(y) + g(y)$ ).

Remark 3:

1) W.V. Petryshyn [7] proves Theorem 2 in the case of a reflexive Banach space $(E, \|\ \|)$ and the weak topology for $E$ (satisfying all conditions of Theorem 2) for a subset $X \subset E$ additionally satisfying $int(X) \neq \emptyset$ (degree method).

2) In the case of a conjugate space $(E, \|\ \|)$ and the weak* topology for $E$ , a $\mathcal{T}$-compact convex subset of

$E$ need not be $\mathcal{T}$ - (sequentially compact). This, however, is true, if $(E, \| \|)$ is strongly separable ([10], p.209).

3) The Krasnoselski condition ($K_1$) is very restrictive, as the following simple example shows: Let $E := \mathbb{R}$ (absolute value norm), $X := [0,1]$; $f, g : X \longrightarrow E$ defined by $f(x) := \frac{1}{2}x$, $g(x) := 1 - \frac{1}{2}x$. Then $(1,0) \in X \times X$ but $f(1) + g(0) = \frac{3}{2} \notin X$. In the case of a Banach contraction $f$ and a compact $g$ and a closed, bounded (strongly), convex subset $X \subset E$ ($K_1$) can be weakened to "$(f+g)(X) \subset X$" ([1],[8]). In our situation this could be done also (see the proof of Theorem 4), if (i) $Id - f$ is demiclosed [8], or (ii) $(Id - f)(X)$ is closed, or (iii) $(Id - f - g)(X)$ is closed, or (iv) If $0 \in \overline{(Id - f - g)(X)}^{strong}$ then $0 \in (Id - f - g)(X)$.

With the same method employed in Theorem 2 - now using Corollary 1 - we can prove

<u>Theorem 3</u>: Let $(E, \| \|)$ be a normed linear space and suppose $\mathcal{T}$ is a Hausdorff topology for $E$, such that

(i) $(E, \mathcal{T})$ is a topological linear space,

(ii) $\bigwedge_{S \subset E}$ $S$ convex $\wedge$ $S$ $\mathcal{T}$-compact $\Longrightarrow$ $S$ is norm-bounded,

(iii) $\bigwedge_{x \in E} \bigwedge_{r \geq 0} B(x,r) := \{y \mid y \in E \wedge \|x-y\| \leq r\} \Longrightarrow B(x,r)$ is $\mathcal{T}$-closed,

(iv) The norm topology for $E$ is finer than $\mathcal{T}$.

Let $R \geq 0$ and suppose $B(0,R)$ is $\mathcal{T}$-compact and $\mathcal{T}$ - (sequentially compact) and $f := B(0,R) \longrightarrow E$ is a generalized contraction, let

$g : [X, \mathcal{T}] \longrightarrow [E, \| \|]$ be sequentially continuous, such that

$(K_2)$ $\bigwedge_{x,y \in E} \|x\| = R \wedge \|y\| \leq R \Rightarrow f(x) + g(y) \in B(0,R)$ .

<u>Then</u> $f + g$ has a fixed point.

<u>Remark 4:</u>

W.V. Petryshyn provës Theorem 3 in [7] in the case of a reflexive Banach space and the weak topology (see Remark 2).

The method developed in [3] yields

<u>Lemma 1</u>: Let $(E, \| \|)$ be a reflexive Banach space and suppose $X$ is a nonvoid, closed, bounded, convex subset of $E$ ; let $f : E \longrightarrow E$ be a uniformly generalized contraction with respect to $X$ and $\{x_n\} \in X^{\mathbb{N}}$ such that $\lim_{n \to \infty} \{x_n - f(x_n)\} = 0$ (strongly).

<u>Then</u> (a) $f$ has a unique fixed point $x_0 \in X$ ,

(b) $\lim_{n \to \infty} \{x_n\} = x_0$ (strongly).

<u>Proof</u>: See [3], proof of Theorem 2.

As a corollary of Lemma 1 we obtain

<u>Lemma 2</u>: Let $(E, \| \|)$ be a reflexive Banach space and suppose $X$ is a nonvoid, closed, bounded, convex subset of $E$ ; let $f : E \longrightarrow E$ be a uniformly generalized contraction with respect to $X$ and let $\{x_n\} \in X^{\mathbb{N}}$ and $y \in E$ such that $\lim_{n \to \infty} \{x_n - f(x_n)\} = y$ (strongly).

<u>Then</u> (a) There is a unique $x_1 \in X$ such that $x_1 - f(x_1) = y$ ,

(b) $\lim_{n \to \infty} \{x_n\} = x_1$ .

Proof: Define $g: E \longrightarrow E$ by $g(x) := f(x) + y$. Then $g$ is a uniformly generalized contraction with respect to $X$ and $\lim_{n\to\infty} \{x_n - g(x_n)\} = 0$ (strongly). Thus, by Lemma 1, there is a unique $x_1 \in X$ such that $g(x_1) = x_1$, i.e. $x_1 - f(x_1) = y$ and $\lim_{n\to\infty} \{x_n\} = x_1$ (strongly).

Theorem 4: Let $(E, \|\ \|)$ be a reflexive Banach space and suppose $X$ is a nonvoid, closed, bounded, convex subset of $E$; let $f: E \longrightarrow E$ be a uniformly generalized contraction with respect to $X$ and let $g: X \longrightarrow E$ be compact such that $(f + g)(X) \subset X$.
Then $f + g$ has a fixed point.

Proof: Without loss of generality we may assume $0 \in X$. Let $\{\lambda_n\} \in (0,1)^{\mathbb{N}}$ with $\lim_{n\to\infty} \{\lambda_n\} = 1$. We define $f_n := \lambda_n f$, $g_n := \lambda_n g$ for $n \in \mathbb{N}$ and we have $(f_n + g_n)(X) \subset X$. Because of $\|f_n(x) - f_n(y)\| \leq \lambda_n \alpha(x)\|x - y\| \leq \lambda_n \|x - y\|$ and $g_n$ being compact, there is a sequence $\{x_n\} \in X^{\mathbb{N}}$ such that $f_n(x_n) + g_n(x_n) = x_n$ (see [1],[8]). Because of $g$'s compactness there exists a subsequence $\{x'_n\} \in X^{\mathbb{N}}$ of $\{x_n\}$ and $y \in E$ such that $\lim_{n\to\infty} \{g(x'_n)\} = y$ (strongly). Now we have for $n \in \mathbb{N}$: $x'_n - f(x'_n) - g(x'_n) = (\lambda'_n - 1)(f(x'_n) + g(x'_n))$. The boundedness of $X$ implies $\lim_{n\to\infty} \{x'_n - f(x'_n)\} = y$ (strongly). By Lemma 2 we have a $x_1 \in X$ with $x_1 - f(x_1) = y$ and $\lim_{n\to\infty} \{x'_n\} = x_1$ (strongly). Finally the continuity of $g$ induces $\lim_{n\to\infty} \{g(x'_n)\} = g(x_1)$ such that $y = g(x_1)$: We have $x_1 - f(x_1) = g(x_1)$, i.e. $f(x_1) + g(x_1) = x_1$, q.e.d.

The same method used in the proof of Theorem 4 yields

Theorem 5: Let $(E, \| \|)$ be a reflexive Banach space and suppose $X$ is a closed, bounded, convex subset of $E$ and $x_0 \in int(X)$; let $f: E \to E$ be a uniformly generalized contraction with respect to $X$ and $g: X \to E$ be such that

$$(K_3) \bigwedge_{x,y \in E} \bigwedge_{\lambda \in R} x \in brd(X) \wedge (f+g)(x) = \lambda x + (1-\lambda)x_0 \Longrightarrow \lambda \leq 1 .$$

Then $f + g$ has a fixed point.

Remark 5:

Theorem 5 is proved by W.A. Kirk in [3] for $x_0 = 0$ (using a method of F.E. Browder [2]) and by W.V. Petryshyn in [7] (degree method).

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(Oblatum 27.7.1972)