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A NOTE ON COMPATIBLE REFLEXIVE RELATIONS ON QUASIGROUPS

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Basic definitions used in this paper can be found in [1] or [2].

A relation ϱ on a groupoid G will be called compatible if for all $a, b, c, d \in G$:

$$(a \varrho b \text{ et } c \varrho d) \Longrightarrow ac \varrho bd .$$

A reflexive relation ϱ on G will be called semicompatible if for all $a, b, c \in G$:

$$a \varrho b \Longrightarrow (ac \varrho bc \text{ et } ca \varrho cb) .$$

A relation ϱ on G is called normal if for all $a, b, c, d \in G$:

$$(ac \varrho bd \text{ et } (a \varrho b \text{ vel } c \varrho d)) \Longrightarrow (a \varrho b \text{ et } c \varrho d) .$$

A reflexive relation ϱ on G is called seminormal if for all $a, b, c \in G$:

$$(ac \varrho bc \text{ vel } ca \varrho cb) \Longrightarrow a \varrho b .$$

The following lemma is evident.

Lemma 1. Let G be a groupoid and ϱ a reflexive relation on G . Then:

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(i) if ϱ is compatible then ϱ is semicompatible;

(ii) if ϱ is semicompatible and transitive then ϱ is compatible;

(iii) if ϱ is normal then ϱ is seminormal;

(iv) if ϱ is seminormal, semicompatible, transitive and symmetric, then ϱ is normal and compatible.

Theorem 1. Let G be a commutative groupoid and ϱ a reflexive relation on G. Then:

(i) if ϱ is normal, then ϱ is symmetric;

(ii) if ϱ is compatible and seminormal, then ϱ is transitive;

(iii) if ϱ is compatible and normal, then ϱ is a normal congruence relation.

Proof. (i) Let $a, b \in G$ and $a \varrho b$. We have $ab \varrho ab$, $ab = ba$. Hence $b \varrho a$ (since ϱ is normal).

(ii) Let $a, b, c \in G$ be such that $a \varrho b$ and $b \varrho c$. Hence $ab \varrho bc$. But $bc = cb$. Thus $a \varrho c$.

The statement (iii) follows from (i) and (ii).

Theorem 2. Let Q be a division groupoid and ϱ a reflexive normal compatible relation on Q. Then ϱ is a normal congruence relation on Q.

Proof. At first we shall prove that ϱ is transitive. Let $a, b, c \in Q$ be such that $a \varrho b$ and $b \varrho c$. There are $x, y \in Q$ such that $bx = ay = a$. We have $a \varphi a$, that is $ay \varphi bx$. Hence $y \varphi x$. Further, we have $bx \varphi cx$, hence $ay \varphi cx$. But $y \varphi x$. Therefore $a \varphi c$. Now we shall prove that φ is symmetric. Let $a, b \in Q$ and let $a \varphi b$. There are x, y, z such that $ax = by = b$, $bz = a$. Thus we can write $bz \varphi by$. Hence $z \varphi y$, and hence, $az \varphi by$. Therefore $az \varphi b$. But $b = ax$. Hence $az \varphi ax$. Hence $z \varphi x$. Further $a \varphi b$, which means $bz \varphi ax$. Since $z \varphi x$, we get $b \varphi a$.

In the remaining part of this paper we shall prove that every cancellation groupoid can be imbedded in a quasigroup, every semicompatible and reflexive relation of which is seminormal. Such a quasigroup will be called a $\mathcal{N}$-quasigroup. It is evident that every $\mathcal{N}$-groupoid is a cancellation groupoid and hence its every subgroupoid is a cancellation groupoid.

Theorem 3. Let G be a $\mathcal{N}$-groupoid. Then every semicompatible equivalence relation on G is a normal congruence relation. Further, every semicompatible ordering on G is a seminormal compatible ordering.

Proof: By Lemma 1.

Lemma 2. Let Q be a quasigroup. Then there are a quasigroup $\tilde{Q}$ and mappings α, β of Q into $\tilde{Q}$ such that Q is a subquasigroup of $\tilde{Q}$ and for all $x, y \in Q$ it holds:

$$\alpha(x)(\beta(x)(xy)) = y .$$

__Proof__. Select for every $a, b, c, d \in Q$ different symbols $\sigma(a), \tau(b), \varphi(c, d)$. Let R be the set consisting of all elements of Q and of all symbols $\sigma(a)$, $\tau(b)$, $\varphi(c,d)$. On the set R, we shall define a partial binary operation $*$. Let $a, b \in R$. Then $a * b$ is defined only in the following cases:

(i) $a, b \in Q$. Then $a * b = ab$.

(ii) There is $c \in Q$ such that $a = \sigma(c)$ and $b \in Q$. Then $a * b = \varphi(c, b)$.

(iii) There are $c, d \in Q$ such that $a = \tau(c)$, $b = \varphi(c, d)$. Then $a * b = e$, where $e \in Q$ such that $ce = d$.

$R(*)$ is a halfgroupoid and Q is a subquasigroup of $R(*)$. We shall prove that $R(*)$ is a cancellation halfgroupoid. At first the left-cancellation law.

Let $a, b, c \in R(*)$, let $a * b, a * c$ be defined and $a * b = a * c$. Such cases can arise:

(i) $a \in Q$. Then necessarily $b, c \in Q$ and $a * b = ab = a * c = ac$. Hence $b = c$.

(ii) There is $d \in Q$ such that $a = \sigma(d)$. Hence $b, c \in Q$ and $a * b = \varphi(d, b) = a * c = \varphi(d, c)$. Therefore $b = c$.

(iii) There is $d \in Q$ such that $a = \tau(d)$. Hence there are $e, f \in Q$ such that $b = \varphi(d, e)$, $c = \varphi(d, f)$. Then $a * b = g = a * c = h$, where $dg = e$, $dh = f$. But $g = h$, hence $e = f$, and hence, $b = c$.

Now the right cancellation law. Let $a, b, c \in R(*)$

and $b * a = c * a$. We must discuss the following cases:

(i) $a, b, c \in Q$. Then $b * a = ba = c * a = ca$. Hence $b = c$.

(ii) $a \in Q$ and there are $d, e \in Q$ such that $b = \sigma(d)$, $c = \sigma(e)$. Then $b * a = \varphi(d,a) = c * a = \varphi(e,a)$. Therefore $d = e$, hence $b = c$.

(iii) There are $d, e \in Q$ such that $a = \varphi(d, e)$. Then necessarily $b = \tau(d) = c$.

It is well known that every cancellation halfgroupoid can be imbedded in a quasigroup. (See R.H. Bruck: A survey of binary systems, Springer-Verlag, 1966.) Hence there is a quasigroup $\tilde{Q}$ such that $R(*)$ is a subhalfgroupoid of $\tilde{Q}$, If x, y are arbitrary elements of Q then

$$\tau(x)(\sigma(x)(xy)) = \tau(x) * (\sigma(x) * xy) =$$

$$= \tau(x) * \varphi(x, xy) = y .$$

Now it is sufficient to put $\alpha(x) = \tau(x)$, $\beta(x) = \sigma(x)$.

Lemma 3. Let Q be a quasigroup. Then there are a quasigroup $\bar{Q}$ and mappings α, β of Q into $\bar{Q}$ such that Q is a subquasigroup of $\bar{Q}$ and for every $x, y \in Q$ it holds:

$$((yx)\beta(x))\alpha(x) = y .$$

Proof. The proof is dual to that of Lemma 2.

Theorem 4. Any cancellation groupoid can be imbedded in an $\mathcal{N}$-quasigroup.

Proof. Let Q be a given groupoid. Since Q can be imbedded in a quasigroup, we can presume without loss of generality that Q is a quasigroup. Put $Q = Q_0$, $Q_i = \tilde{Q}_{i-1}$ for all odd $i \geq 1$, $Q_i = \bar{Q}_{i-1}$ for all even $i \geq 2$ ($\tilde{Q}_i, \bar{Q}_i$ in the sense of Lemmas 2,3). We have $Q = Q_0 \subseteq$ $\subseteq Q_1 \subseteq Q_2 \subseteq \dots$. There is a quasigroup P such that $P = \bigcup_{i=0}^{\infty} Q_i$ and Q_i are subquasigroups of P. Be φ a semicompatible reflexive relation on P. Let $a, b, c \in P$ and let $ab \,\varphi\, ac$. There is an even $i \geq 2$ such that $a, b, c \in Q_i$. But $Q_{i+1} = \tilde{Q}_i$. Hence there are mappings α_i, β_i of Q_i into Q_{i+1} such that $\alpha_i(x)(\beta_i(x)(xy)) = y$ for all $x, y \in Q_i$. Hence we have $c = \alpha_i(a)(\beta_i(a)(ac))$, $b = \alpha_i(a)(\beta_i(a)(ab))$. But φ is semicompatible. Thus

$$\alpha_i(a)(\beta_i(a)(ab)) \,\varphi\, \alpha_i(a)(\beta_i(a)(ac)) .$$

Hence $b \,\varphi\, c$. Similarly if $ba \,\varphi\, ca$. Therefore P is an $\mathcal{N}$-quasigroup.

References

[1] R.H. BRUCK: A Survey of Binary Systems, Springer-Verlag, 1966.

[2] V.D. BELOUSOV: Osnovy teorii kvazigrupp i lup, Nauka, Moskva, 1967.

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