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THE TOTALLY SIMPLE QUASIGROUPS

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Basic definitions used in this paper can be found in [1] or [2].

Let Q be a quasigroup and φ a relation on the set Q .

We shall define the following conditions:

- (1) $\forall a, b, c \in Q, a \varphi b \Rightarrow a c \varphi b c$.
- (2) $\forall a, b, c \in Q, a \varphi b \Rightarrow c a \varphi c b$.
- (3) $\forall a, b, c \in Q, a c \varphi b c \Rightarrow a \varphi b$.
- (4) $\forall a, b, c \in Q, c a \varphi c b \Rightarrow a \varphi b$.
- (5) $\forall a, b, c, d \in Q, (ac = bd \text{ et } c \varphi d) \Rightarrow a \varphi b$.
- (6) $\forall a, b, c, d \in Q, (ac = bd \text{ et } a \varphi b) \Rightarrow c \varphi d$.
- (7) $\forall a, b, c, d \in Q, (a \varphi b \text{ et } c \varphi d) \Rightarrow ac \varphi bd$.
- (8) $\forall a, b, c, d \in Q, (ac \varphi bd \text{ et } c \varphi d) \Rightarrow a \varphi b$.
- (9) $\forall a, b, c, d \in Q, (ac \varphi bd \text{ et } a \varphi b) \Rightarrow c \varphi d$.

Let M be a set. By σ_M , we shall denote the set of all pairs (a, a) , $a \in M$. Further denote $\pi_M = M \times M$ and $\sigma_M = \pi_M \setminus \sigma_M$; that is, σ_M consists of all ordered pairs (a, b) , where $a, b \in M$, $a \neq b$. Hence

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σ_M, π_M and σ_M are relations on the set M .

If Q is a quasigroup then it is evident that the relations $\sigma_Q, \pi_Q, \sigma_Q$ satisfy the conditions (1),(2),(3), (4),(5),(6). Moreover, σ_Q and π_Q satisfy (7),(8),(9). Every equivalence relation ρ on Q that satisfies (7), (8),(9) is called a normal congruence relation.

A quasigroup Q is called simple if every its normal congruence relation is equal to one of the relations σ_Q, π_Q . A quasigroup Q is called totally simple if every its relation ρ that satisfies at least one of the conditions (1),(2),(3),(4),(5),(6) is equal to one of the relations $\sigma_Q, \pi_Q, \sigma_Q$. Evidently, every totally simple quasigroup is simple. In this paper we shall prove that every quasigroup can be imbedded in a totally simple quasigroup.

Lemma 1. Let $Q(\ast)$ be the right inverse quasigroup of a quasigroup Q . Let ρ be a relation on the set Q . Then:

- (i) ρ satisfies (1) on Q if and only if ρ satisfies (6) on $Q(\ast)$.
- (ii) ρ satisfies (2) on Q if and only if ρ satisfies (4) on $Q(\ast)$.
- (iii) ρ satisfies (3) on Q if and only if ρ satisfies (5) on $Q(\ast)$.

Proof. (i) Let ρ satisfy (1) on Q and let $a, b, c, d \in Q$ be such that $a \ast c = b \ast d$ and $a \rho b$. Put $x = a \ast c$. Hence $ax = c, bx = d$. Since $a \rho b$, $ax \rho bx$. Thus ρ satisfies (6) on $Q(\ast)$. Again,

let φ satisfy (6) on $Q(*)$ and $a, b, c \in Q$ be such that $a \varphi b$. There are $x, y \in Q$ such that $a * x = b * y = c$. Since φ satisfies (6), $x \varphi y$. But $x = ac, y = bc$. The proof of (ii) and (iii) is similar to that of (i).

Lemma 2. Let $Q(o)$ be the left inverse quasigroup of a quasigroup Q . Let φ be a relation on the set Q . Then:
 (i) φ satisfies (1) on Q if and only if φ satisfies (3) on $Q(o)$.
 (ii) φ satisfies (2) on Q if and only if φ satisfies (5) on $Q(o)$.
 (iii) φ satisfies (4) on Q if and only if φ satisfies (6) on $Q(o)$.

Proof. Similarly as for Lemma 1.

Lemma 3. For every quasigroup Q there is a quasigroup \bar{Q} such that Q is a subquasigroup of \bar{Q} and if $a, b, c, d \in Q, a \neq b, c \neq d$, then there is $x \in \bar{Q}$ such that $ax \cdot x = c, bx \cdot x = d$.

Proof. For every $a, b, c, d \in Q, a \neq b, c \neq d$, select (pair-wise different) symbols $x(a, b, c, d), y(a, b, c, d), u(a, b, c, d)$. Let H be the set consisting of all these symbols and of all elements of Q . We shall define a partial binary operation $*$ on H . Let $m, n \in H$. Then $m * n$ is defined only in the following cases:

(i) If $m, n \in Q$. Then $m * n = mn$.

(ii) If $m \in Q$ and if there are $a, b, c, d \in Q, a \neq b, c \neq d$, such that $m = a, n = x(a, b, c, d)$. Then $m * n = y(a, b, c, d)$.

(iii) If $m \in Q$ and if there are $a, b, c, d \in Q, a \neq b, c \neq d$, such that $m = b, n = x(a, b, c, d)$. Then $m * n = u(a, b, c, d)$.

(iv) If there are $a, b, c, d \in Q, a \neq b, c \neq d$, such that $m = y(a, b, c, d), n = x(a, b, c, d)$. Then $m * n = c$.

(v) If there are $a, b, c, d \in Q, a \neq b, c \neq d$, such that $m = u(a, b, c, d), n = x(a, b, c, d)$. Then $m * n = d$.

It is easy to show that $H(*)$ is a cancellation halfgroupoid. Hence $H(*)$ can be imbedded in a quasigroup $\overline{Q}(*)$. Evidently, Q is a subquasigroup in $\overline{Q}(*)$. Let $a, b, c, d \in Q, a \neq b, c \neq d$. Put $x = x(a, b, c, d)$. Then $(a * x) * x = c, (b * x) * x = d$.

Theorem 1. Let Q be a quasigroup. Then Q can be imbedded in a quasigroup P having the following property: If $a, b, c, d \in P, a \neq b, c \neq d$, then there is $x \in P$ such that $a * x * x = c, b * x * x = d$.

Proof. Put $Q_1 = Q$ and $Q_{i+1} = Q_i$ for $i = 1, 2, 3, \dots$ (\overline{Q}_i by Lemma 3). We have $Q_1 \subseteq Q_2 \subseteq Q_3 \subseteq \dots$. Hence there is a quasigroup P such that $P = \bigcup_{i=1}^{\infty} Q_i$. Let $a, b, c, d \in P, a \neq b, c \neq d$. There is i such that $a, b, c, d \in Q_i$. By Lemma 3, there

exists $x \in Q_{i+1}$ such that $ax \cdot x = c$, $bx \cdot x = d$.

Let $i = 1, 2, 3, 4, 5, 6$. A quasigroup Q will be called an (i)-quasigroup if every its relation satisfying the condition (i) is equal to one of σ_Q , π_Q , θ_Q .

Lemma 4. Every quasigroup Q can be imbedded in an (i)-quasigroup $Q^{(i)}$ for every $i = 1, 2, 3, 4, 5, 6$.

Proof. First for $i = 1$. By Theorem 1, there is a quasigroup P having the following property:

Q is a subquasigroup of P and for every $a, b, c, d \in P$, $a \neq b$, $c = d$, there is $x \in P$ such that $ax \cdot x = c$, $bx \cdot x = d$. Let φ be a relation on P and let φ satisfy (1). Let $\varphi \neq \sigma_P$. Since φ satisfies (1) there are $a, b \in P$ such that $a \neq b$ and $a \varphi b$. Let $c, d \in P$, $c \neq d$, be arbitrary. There is $x \in P$ such that $ax \cdot x = c$, $bx \cdot x = d$. Since φ satisfies (1), $c \varphi d$. Hence $\theta_P \subseteq \varphi$. Further, let $\varphi \neq \theta_P$. Hence there is $a \in P$ such that $a \varphi a$. Let $b \in P$ be arbitrary and $c \in P$ be such that $ac = b$. Since $a \varphi a$, $ac \varphi ac$. Thus $b \varphi b$ and $\varphi = \pi_P$. Now it is sufficient to put $P = Q^{(1)}$.

Now for $i = 6$. Let $Q(\ast)$ be the right inverse quasigroup of Q . There is a (1)-quasigroup $P(\ast)$ such that $Q(\ast)$ is a subquasigroup of $P(\ast)$. Let P be the right inverse quasigroup of $P(\ast)$. Evidently, Q is a subquasigroup in P . Let φ be a relation on the set P and let φ satisfy (6) on P . By Lemma 1, φ satisfies (1) on $P(\ast)$. Hence φ is equal to one of σ_P , θ_P , π_P .

Thus P is a (6)-quasigroup.

For $i = 2, 3, 4, 5$ similarly as for $i = 6$ by Lemmas 1, 2.

Theorem 2. Every quasigroup can be imbedded in a totally simple quasigroup.

Proof. Let Q be the given quasigroup. Let $\alpha(i)$, for $i = 1, 2, 3, \dots$ be such a number that $1 \leq \alpha(i) \leq 6$ and $\alpha(i) \equiv i \pmod{6}$. Put $Q_1 = Q$ and $Q_{i+1} = Q_i^{(\alpha(i))}$ for $i = 1, 2, 3, \dots$

($Q_i^{(\alpha(i))}$ by Lemma 4). Thus $Q_1 \subseteq Q_2 \subseteq Q_3 \subseteq \dots$. Hence there is a quasigroup P such that $P = \bigcup_{i=1}^{\infty} Q_i$. It is easy to show that P is totally simple.

R e f e r e n c e s

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