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FRATTINIAN CONSTRUCTIONS

Jarmila LISÁ, Praha

1. Introduction. In accordance with [1] and [3], we define the Frattini sublattice of a lattice as the intersection of all its maximal sublattices and we denote it by $\Phi(L)$; if there is no maximal sublattice, we define $\Phi(L) = L$ (by a maximal sublattice, here a maximal proper one is meant). So we have the analogue of Frattini's construction, very well known in groups ([2], p.156),

We use these symbols and assumptions:

$\Cap$ (resp. $\Cup$) signifies the symbol for the intersection (resp. for the union) of sets. Further, $[a, b, \dots]$ (resp. $\{a, b, \dots\}_L$) denotes the set consisting of $a, b, \dots$ (resp. the sublattice generated by the set $[a, b, \dots]$).

We assume that $\emptyset$ is a lattice and that the axiom of choice holds.

We shall often use the following assertions:

Let L be a lattice. Then

(i) $\{\mathcal{I}r(L)\}_L \setminus \mathcal{I}r(L) \subseteq \Phi(L) \subseteq L(\cup) \Cup L(\cap)$

(cf.[3], Lemma 2; $\mathcal{I}r(L)$ means the set of all irredu-

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cible elements of L , $L(\cup)$, or resp. $L(\cap)$, means the set of all $\cup$-reducible, or $\cap$-reducible elements of L).

(ii) $\Phi(L) = [x \mid (x \in L) \& (\forall T \subseteq L, \{T, x\}_L = L \Rightarrow \{T\}_L = L)]$ (cf.[2], p.156).

2. Direct product

Let L_1, L_2 be lattices, $L_1 \times L_2$ be their direct product. Generally, it is not true that $\Phi(L_1 \times L_2) = \Phi(L_1) \times \Phi(L_2)$. We shall introduce some conditions which permit to go over to the decomposition of the Frattini sublattice formed for the direct product of lattices.

Theorem 1. (a) Let L_1, L_2 be lattices and let any maximal sublattice M of $L_1 \times L_2$ be of the form $M = A \times B$ where A is a sublattice of L_1, B is a sublattice of L_2. Then $\Phi(L_1) \times \Phi(L_2) \subseteq \Phi(L_1 \times L_2)$.

(b) Let for any maximal sublattice M_1 of L_1 and for any maximal sublattice M_2 of L_2 the lattices $M_1 \times L_2$ and $L_1 \times M_2$ be maximal sublattices of $L_1 \times L_2$. Then

$$\Phi(L_1) \times \Phi(L_2) \supseteq \Phi(L_1 \times L_2) .$$

Proof. 1) Let $c = (c_1, c_2) \in \Phi(L_1) \times \Phi(L_2)$ and $c \notin (L_1 \times L_2)$. By (a) there exists a maximal sublattice M of $L_1 \times L_2$, $c \notin M$ such that $M = A \times B$. Since M is maximal in $L = L_1 \times L_2$, it must be either $M = A \times L_2$ where A is a maximal sublattice of L_1, or $M = L_1 \times B$, B being a maximal sublattice of L_2.

Let $M = A \times L_2$ where A is a maximal sublattice of L_1. Since $c \notin M = A \times L_2$, $c_1 \notin A$, $c_1 \notin \Phi(L_1)$ - a contradiction.

2) Let us suppose (b). If $c_1 \notin \Phi(L_1)$, then there exists a maximal sublattice M_1 of L_1 such that $c_1 \notin M_1$. By assumption, $M_1 \times L_2$ is a maximal sublattice of L. For any element h_2 of L_2 we have $(c_1, h_2) \notin M_1 \times L_2$ and it follows that $(c_1, h_2) \notin \Phi(L)$. Thus $\Phi(L) \subseteq \Phi(L_1) \times \Phi(L_2)$.

Corollary. If the conditions (a) and (b) hold, then
$\Phi(L_1) \times \Phi(L_2) = \Phi(L_1 \times L_2)$.

Definition. Let L be a lattice. We shall say that L satisfies the X-condition for the element b, if there exists a maximal sublattice K of L which does not contain b and which contains some b_1, b_2 such that $b_1 < b < b_2$.

L satisfies the X-condition, if L satisfies the X-condition for any element of $L \setminus \Phi(L)$.

Lemma 1. Let L_1, L_2 be lattices, $b \in L_2$, L_2 satisfying the X-condition for the element b. Then
$\Phi(L_1 \times L_2) \subseteq L_1 \times (L_2 \setminus [b])$.

Proof. $L_1 = \emptyset$ - trivial.

We assume $L_1 \neq \emptyset$; K, b_1, b_2 are used in the same sense as in the definition. We shall show that $(x, y) \notin \Phi(L_1 \times L_2)$ whenever $(x, y) \notin L_1 \times (L_2 \setminus [b])$. It is sufficient to show that there exists a proper sublattice T of $L_1 \times L_2$ having the property $\{T, (x, y)\}_{L_1 \times L_2} = L_1 \times L_2$. In our case we can take $T = L_1 \times K$ (clearly, $L_1 \times K \subsetneqq L_1 \times L_2$), $(x, y) = (a, b)$ for arbitrary element a of L_1. We shall easily verify that $\{L_1 \times K, (a, b)\}_{L_1 \times L_2} = L_1 \times L_2$: Indeed, let $(k, u) \in L_1 \times L_2$ be arbitrary. As K is a maximal sublattice of L_2 and $b \notin K$, then $u = f(x_1, \dots, x_m)$ where f is a lattice polynomial in L_2 and $x_1, \dots, x_m \in K \cup [b]$. Then $(k, u) = f'(y_1, \dots, y_m)$ where f' is the same lattice polynomial as f, but in $L_1 \times L_2$, $y_i = (k, x_i)$, $i = 1, \dots, m$. If $x_i \neq b$, then clearly $(k, x_i) \in L_1 \times K$; if $x_i = b$, then $(k, b) = ((k, b_1) \cup (a, b)) \cap (k, b_2)$, i.e. $(k, b) \in \{L_1 \times K, (a, b)\}_{L_1 \times L_2}$; so $(k, u) \in \{L_1 \times K, (a, b)\}_{L_1 \times L_2}$.

__Theorem 2.__ Let L_1, L_2 be lattices, let L_2 satisfy the X-condition. Then $\Phi(L_1 \times L_2) \subseteq L_1 \times \Phi(L_2)$.

__Proof__ follows by Lemma 1.

An immediate consequence of Theorem 2 is the following:

__Corollary 1.__ Let L_2 be a chain without 0 and 1, L_1 being an arbitrary lattice. Then $\Phi(L_1 \times L_2) = \emptyset$.

__Corollary 2.__ For an arbitrary lattice L_1 and any distributive lattice L_2 without 0 and 1

$\Phi(L_1 \times L_2) \subseteq L_1 \times \Phi(L_2)$.

__Proof.__ We shall show that L_2 satisfies the X-condition. Let us suppose that this is not true, i.e., that there exists an element $b \in L_2 \setminus \Phi(L_2)$ such that for any maximal sublattice K of L_2 which does not contain b, it is either $K \subseteq L_2 \setminus [b)$, or $K \subseteq L_2 \setminus (b]$. Say that e.g. $K \subseteq L_2 \setminus [b)$.

In this case there is clearly an element b_1 of L_2 such that $\mathscr{b} < \mathscr{b}_1$. Since $\mathscr{b}_1 \notin K, \{K, \mathscr{b}_1\}_{L_2} = L_2$, and hence $\mathscr{b} \in \{K, \mathscr{b}_1\}_{L_2}$. By distributivity of L_2 , one of the following cases is necessarily true:

1) $\mathscr{b} = \mathscr{b}_1 \cup \mathscr{k}$ for some $\mathscr{k} \in K$,

2) $\mathscr{b} = \mathscr{b}_1 \cap \mathscr{k}$ for some $\mathscr{k} \in K$,

3) $\mathscr{b} = (\mathscr{b}_1 \cup \mathscr{k}) \cap \mathscr{h}$ for some $\mathscr{k}$, $\mathscr{h} \in K$

and it is easy to check that we obtain a contradiction in each of these cases.

By $A \times B \cong B \times A$, it is immediate that the following assertion holds:

Theorem 2'. Let L_1 be a lattice satisfying the $\mathcal{X}$ - condition, L_2 be an arbitrary lattice. Then

$$\Phi(L_1 \times L_2) \subseteq \Phi(L_1) \times L_2 .$$

It is possible to obtain similar results from Corollaries 1 and 2.

3. L -sum

Let L be a lattice with the partial ordering $\leq_L$ and lattice operations $\cup_L$, $\cap_L$, $A \subseteq \mathcal{I}rr(L)$ a possibly empty set, $\mathcal{U} = [L_a \mid a \in A]$ a family of pairwise disjoint lattices which are all disjoint with L ; if $a \in A$, the partial ordering in L_a is denoted by $\leq_a$, the lattice operations are denoted by $\cup_a$, $\cap_a$.

Denote $K = (L \setminus A) \uplus \biguplus_{a \in A} L_a$ and define a binary relation $\leq$ in K :

$$x, y \in K,\ x \leq y \iff \begin{cases} x, y \in L, \text{ then } x \leq_L y ; \\ x, y \in L_a,\ a \in A,\ x \leq_a y ; \\ x \in L_a, y \in L_b, a, b \in A,\ a \neq b,\ a \leq_L b ; \\ x \in L_a, y \in L \setminus A,\ a \in A,\ a \leq_L y ; \\ x \in L \setminus A,\ y \in L_a,\ a \in A,\ x \leq_L a . \end{cases}$$

The relation $\leq$ is a partial ordering. K is even a lattice with operations $\cup, \cap$; we shall describe $x \cup y, x \cap y$ for $x \parallel y$:

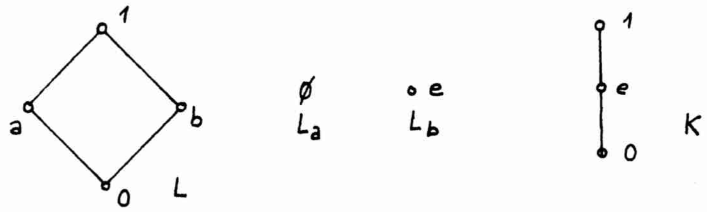
$$x, y \in K,\ x \parallel y ; \left\{ \begin{matrix} x \cup y \\ \\ x \cap y \end{matrix} \right\} = \begin{cases} \left\{ \begin{matrix} x \cup_L y \\ x \cap_L y \end{matrix} \right\} & x, y \in L ; \\ \left\{ \begin{matrix} x \cup_a y \\ x \cap_a y \end{matrix} \right\} & x, y \in L_a,\ a \in A ; \\ \left\{ \begin{matrix} a \cup_L b \\ a \cap_L b \end{matrix} \right\} & x \in L_a,\ y \in L_b, a, b \in A, a \neq b ; \\ \left\{ \begin{matrix} a \cup_L y \\ a \cap_L y \end{matrix} \right\} & x \in L_a, a \in A,\ y \in L \setminus A . \end{cases}$$

We shall call the lattice K the L -sum of the family $\mathcal{Ol}$ and we denote K by $\Sigma_L (L_a \mid a \in A)$. It represents a generalization of L -sum defined in [3].

Now, let L be a lattice with $\mathcal{I}rr(L) \neq \emptyset$; we can ask if $\Phi(\Sigma_L (L_a \mid a \in \mathcal{I}rr(L))) = \Phi(L) \uplus \biguplus_{a \in A} \Phi(L_a)$ (cf. [3], Lemma 2).

As we assume that $\emptyset$ is a lattice, this is not in general true, which can be demonstrated by Fig.1.

Fig. 1



$\Phi(L) = [0,1]$ $\quad$ K is the L-sum of L_a, L_b ;
$\Phi(K) = \emptyset$.

However, we can show

<u>Theorem 3</u>. Let L be a lattice, $A \subseteq \mathcal{I}rr(L)$ and let $L_a \neq \emptyset$ be a lattice for all $a \in A$. Then

$$\Phi(\Sigma_L(L_a \mid a \in A)) = \Phi(L) \cup \bigcup_{a \in A} \Phi(L_a) \quad .$$

<u>Proof</u>. Denote $K = \Sigma_L(L_a \mid a \in A)$.

First we shall describe all maximal sublattices of K .

1) Let M be a maximal sublattice of L and $A \subseteq M$. Then $M' = \Sigma_M(L_a \mid a \in A)$ is a maximal sublattice of K by the definition of the binary operations on K .

2) Let N be a maximal sublattice of L_b for some $b \in A$. Then $M = \Sigma_L(L'_a \mid a \in A)$ is again a maximal sublattice of K where for any element $a \in A$, $a \neq b$, there is $L'_a = L_a$ and $L'_b = N$.

3) The maximal sublattice of a different type does not exist: If M is a maximal sublattice of K , let us denote

$L'_a = M \cap L_a$ for $a \in A$,

$B = (M \cap L) \cup [a \mid a \in A, L'_a \neq \emptyset]$, i.e., $\mathcal{M} = \Sigma_B(L'_a \mid a \in A \cap B)$.

Let $x \in K \setminus \mathcal{M}$ be arbitrary, then $\{\mathcal{M}, x\}_K = K$.

a) If $x \in L$, then $\{\mathcal{M}, x\}_K = (\{B, x\}_L \setminus A) \uplus \biguplus_{a \in A \cap B} L'_a$ and this implies B is a maximal sublattice of L and for all $a \in A$ $L_a = L'_a$.

b) If $x \in L_b$ for some $b \in A$, then $\{\mathcal{M}, x\}_K = (\{B, b\}_L \setminus A) \uplus \biguplus_{a \in (A \cap B) \setminus \{b\}} L'_a \uplus \{L'_b, x\}_{L_b}$, therefore for all $a \in A$, $a \neq b$, we have $L'_a = L_a$, L'_b is a maximal sublattice of L_b and $\{B, b\}_L = L$.

In the case a), the maximal sublattice is of the same type as in 1), in the case b) it is of the same type as in 2).

Now we obtain immediately: $\Phi(\Sigma_L(L_a \mid a \in A)) = \Phi(L) \uplus \biguplus_{a \in A} \Phi(L_a)$.

Corollary. Let $\{L_i \mid i \in I\}$ be a family of lattices. Then $\Phi(\underset{i \in I}{+} L_i) = \underset{i \in I}{+} \Phi(L_i)$ (where $+$ denotes the ordinal sum).

Proof. The ordinal sum is a special case of the L - sum for a chain L . In [3] this corollary follows immediately from Lemma 2, but it is true also provided some of the lattices are empty, for $\underset{i \in I}{+} L_i = \underset{i \in I \setminus J}{+} L_i$ where $J = \{j \mid j \in I, L_j = \emptyset\}$.

4. The Frattini hull and some of its properties

Khee-Meng Koh showed in his interesting paper [3] that for each lattice L , $\mathrm{Card}(L) \geq 1$ there exists a lattice K such that $\Phi(K) = L$. Evidently, it is true also for the lattice $L = \emptyset$.

We shall show here a generalization of Khee-Meng Koh's construction, which gives some stronger results.

Let L be a lattice with a partial ordering $\leq_L$, lattice operations $\cup_L$, $\cap_L$ and $\mathrm{Card}(L) \geq 2$, $\mathfrak{A} \subseteq \mathfrak{D} = [(a, b) \mid a, b \in L, a >_L b]$.

We add two new elements $a_1(b)$, $a_2(b)$ to L for all $(a, b) \in \mathfrak{A}$ such that if $(a, b), (c, d) \in \mathfrak{A}$, $(a, b) \neq (c, d)$, supposing $a_1(b), a_2(b), c_1(d), c_2(d)$ pairwise different. We obtain a set

$K = L \uplus [a_i(b) \mid i = 1, 2, (a, b) \in \mathfrak{A}]$. Let us introduce two unary operations: For $x \in K$ we define $\overline{x}$ or $\underline{x}$ in the following way:

1) $\overline{x} = \underline{x} = x$ if $x \in L$;

2) $\overline{x} = a$, $\underline{x} = b$ if $x = a_i(b)$ for some $(a, b) \in \mathfrak{A}$, $i = 1, 2$.

Let us define a binary relation $\leq_K$ on K :

$x, y \in K$, $x \leq_K y \iff$ if $x = y$ or $\overline{x} \leq_L \underline{y}$.

Evidently, $\leq_K$ is a partial ordering in K . K is even a lattice with lattice operations $\cup_K$, $\cap_K$, which are defined as follows: If $x, y \in K$, $x \leq_K y$, then $x \cup_K y = y$, $x \cap_K y = x$;

if $x \parallel y$, then $x \cup_K y = \overline{x} \cup_L \overline{y}$, $x \cap_K y = \underline{x} \cap_L \underline{y}$.

<u>Theorem 4</u>. Let L be a lattice with $\mathrm{Card}(L) \geq 2$. Then there exists a lattice K which satisfies the following claims:

(i) L is embeddable in K , (ii) $L = K(\cup) \uplus K(\cap)$,

(iii) $\Phi(K) = L$.

Proof can be given by the investigations of the lattice K constructed for $\mathcal{A} = [(a, b) \mid (a, b) \in \mathcal{D}$, $a \notin \Phi(L)$ vel $b \notin \Phi(L)]$.

Evidently, the claim (i) is true.

(ii): If $x \in K \setminus L$, then $x \in \mathcal{Irr}(K)$; if $x \in \mathcal{Irr}(L)$, then $x \notin \Phi(L)$, i.e., there exists $y \in L$ such that either $(x, y) \in \mathcal{A}$ or $(y, x) \in \mathcal{A}$. In this case $x \notin \mathcal{Irr}(K)$ and (ii) is also true.

(iii): $K \setminus L = \mathcal{Irr}(K) \subseteq K \setminus \Phi(K)$, i.e., $\Phi(K) \subseteq L$. If $x \in L$, $x \notin \Phi(K)$, then there is a maximal sublattice M of K such that $x \notin M$, but then $x \notin \Phi(L)$ for $x \notin M \cap L$ and $M \cap L$ is a maximal sublattice in L . By the choice of $\mathcal{A}$, $x \in \{\mathcal{Irr}(K)\}_K \setminus \mathcal{Irr}(K) \subseteq \Phi(K)$ - a contradiction.

Remark. 1) If the following supplement (iv) is added to the hypothesis of Theorem 4,

(iv) every proper sublattice of K can be extended to a maximal one,

it is possible to choose $\mathcal{A} = \mathcal{D}$ (cf.[3], Th. 3).

2) Sometimes it is possible to take $\mathcal{A} = [(a, b) \mid a, b \in L, a \succ_L b]$ (for instance, when for each element x of L there exists an element y with $x \succ_L y$ or $y \succ_L x$).

Definition. Let L be a lattice. We shall call the lattice K Frattini $\mathcal{A}$-hull (or only Frattini hull) of L , iff K is formed from L by the introduced construction for this $\mathcal{A}$ and the claims (i),(ii),(iii) are true in K .

__Theorem 5__. Let L be a lattice with $Card(L) > 1$, let K be its Frattini hull. If L has some of the properties

(1) the lattice satisfies the D.C.C.;

(2) the lattice satisfies the A.C.C.;

(3) the lattice is finite;

(4) the lattice is complete;

(5) the lattice is complemented,

then K has the same property.

Proof. 1) Let D.C.C. be true in L, let

(+) $\quad y_1 >_K y_2 >_K \dots >_K y_m >_K \dots$

be a descending chain of elements in K, then

(++) $\quad \underline{y_1} \geq_L \underline{y_2} \geq_L \dots \geq_L \underline{y_m} \geq_L \dots$

is a chain in L ;

$\underline{y_i} = \underline{y_{i+1}}$ iff there exists an element x of L such that

(+++) $\quad y_i = x_j(y_{i+1})$ for $j = 1$ or $j = 2$.

Evidently, there exists a positive integer m such that the chain (++) has just m different elements; according to this and to (+++), the chain (+) does not contain more than $2m$ elements.

The case (2) can be demonstrated similarly.

3) Let L be a finite lattice, then $\mathcal{D}$ is also a finite set and hence K is finite.

(4) Let L be complete and let M be a subset of K . Denote by H the set $[x \in K \mid \forall y \in M \quad y \leq_K x]$.

If (a) $H \cap M \neq \emptyset$, then $\mathrm{Card}\,(H \cap M) = 1$, i.e., $H \cap M = [h]$ and h is the supremum of M in K; if (b) $H \cap M = \emptyset$, denote $\bigcap (H \cap L)$ by h. We shall show that h is the supremum of M in K. Actually, $x \in M \Rightarrow \forall y \in H \quad x \leq_K y \Rightarrow$

$\Rightarrow \forall y \in H \quad \bar{x} \leq_L \underline{y} \Rightarrow \bar{x} \leq_L h \Rightarrow x \leq_K h$;

further, if $z \in K$, $\forall x \in M \quad x \leq_K z$, then:

$\forall x \in M \quad x \leq_K \bar{x} \leq_L \underline{z} \Rightarrow \underline{z} \in H \Rightarrow h \leq_L \underline{z} \Rightarrow h \leq_K z$.

(5) Let L be a complemented lattice, $x \in L$ and let x' denote a complement of the element x.

For $y \in K$ we shall distinguish the following cases:

If (i) $\bar{y} \neq 1$, then $(\bar{y})'$ is a complement of y in K;

if (ii) $\underline{y} \neq 0$, then $(\underline{y})'$ is a complement of y in K;

if (iii) $\bar{y} = 1$ and $\underline{y} = 0$, i.e. $y = 1_1(0)$ or $1_2(0)$, then $1_1(0)$ is a complement for $1_2(0)$ in K.

This completes the proof of Theorem 5.

<u>Remark</u>. Let L be a lattice with $\mathrm{Card}\,(L) \geq \aleph_0$, let K be its Frattini hull. Then $\mathrm{Card}\,(L) = \mathrm{Card}\,(K)$.

<u>Lemma 2</u>. Let L be a lattice with $\mathrm{Card}\,(L) > 2$, K_1 be its Frattini $\mathfrak{D}$-hull, K_2 be its Frattini $\mathfrak{U}$-hull for $\mathfrak{U} \neq \mathfrak{D}$. Then K_1 is not isomorph to K_2.

<u>Corollary</u>. For each lattice L, $\mathrm{Card}\,(L) > 2$, there exist at least two Frattini hulls which are not isomorph.

Let L be a lattice with $\mathrm{Card}\,(L) > 1$. Denote L by $(L)_0$, the Frattini hull of L by $(L)_1$, $((L)_{n-1})_1$ by $(L)_n$ for arbitrary positive integer n, supposing all Frattini hulls constructed in the same way. It means e.g.

that $(L)_1$ is the Frattini $\mathfrak{A}_1$ -hull of L for $\mathfrak{A}_1 = [(a, b) \mid a, b \in L, b \prec_L a]$, $(L)_2$ is the Frattini $\mathfrak{A}_2$ -hull of (L) for $\mathfrak{A}_2 = [(c, d) \mid c, d \in (L)_1, d \prec_{(L)_1} c]$ and so on.

We define $(L)_\infty = \bigcup_{n=0}^{\infty} (L)_n$ as a lattice where the partial ordering is determined by

$x, y \in (L)_\infty$, $x \leq y \iff \exists n$ such that $x, y \in (L)_n$ and $x \leq_{(L)_n} y$.

Let $*$ denote the transitive closure of the following relation ε in $(L)_\infty$:

$x, y \in (L)_\infty, x \varepsilon y \iff \exists n \geq 1, x \in (L)_n, y \in (L)_{n-1}, \bar{x} = y$, or $\underline{x} = y$

($\bar{x}$, $\underline{x}$ mean the elements corresponding to x under the unary operations defined on the Frattini hull of $(L)_{n-1}$).

<u>Theorem 6</u>. Let L be a lattice with $\mathrm{Card}(L) > 1$, let for all $x, y \in L$, $x <_L y$ there exist $x_1, y_1 \in L$ such that $x \prec_L x_1 \leq_L y_1 \prec_L y$ and let each Frattini $\mathfrak{A}_i$ -hull be of this type:

$$\mathfrak{A}_i \subseteq [(a, b) \mid a, b \in (L)_{i-1}, b \prec_{(L)_{i-1}} a] .$$

Then $\Phi((L)_\infty) = (L)_\infty$.

<u>Proof</u>. Let $a \in (L)_\infty \setminus \Phi((L)_\infty)$, i.e., there exists a maximal sublattice M in $(L)_\infty$ such that $a \notin M$. Clearly, $a \in (L)_n$ for some positive integer n and there is some $b \in (L)_n$ such that $a \prec_{(L)_n} b$ or $b \prec_{(L)_n} a$, say $b \prec_{(L)_n} a$, then $a = a_1(b) \cup a_2(b)$ and therefore e.g. $a_1(b) \notin M$. If an element $y \in (L)_\infty$ such that $y * a_2(b)$ or $y = a_2(b)$ is contained in M , then $M \subseteq L \setminus X$ where $X = [a, a_1(b)] \cup [x \mid x \in (L)_\infty, x * a_1(b)]$.

But each element x of $X \setminus [a]$ does not belong to $\{(L \setminus X) \cup [a]\}_{(L)\infty}$, especially, $a_1(b) \notin \{M, a\}_{(L)\infty}$ - a contradiction.
Then $a_2(b)$ and the elements y, $y * a_2(b)$, are not contained in M and we again obtain a contradiction in the same way.

5. Iterations of the Frattini sublattice and the first problem of [3]

Let L be a lattice and α and ordinal number. Denote by $\Phi^0(L)$ the lattice L. We shall proceed by transfinite induction in defining $\Phi^\alpha(L) = \Phi(\Phi^{\alpha-1}(L))$ if $\alpha - 1$ exists and $\Phi^\alpha(L) = \bigcap_{\beta<\alpha} \Phi^\beta(L)$ for α limiting ordinal.

We shall say that K is a <u>submaximal sublattice</u> of L of the order 0 iff $K = L$. We shall call K the <u>submaximal sublattice</u> of L of the order $\alpha + 1$ iff one of the following cases takes place:
Case I. K is a maximal sublattice of a submaximal sublattice of the order α.
Case II. There is no maximal sublattice in every submaximal sublattice of the order α and K is a submaximal sublattice of the order α.

Finally, K is said to be a <u>submaximal sublattice</u> of L of the order α where α is a limiting ordinal iff $K = \bigcap K'$ where K' range over all submaximal sublattices of the orders $\beta < \alpha$.

We denote by $\mathcal{K}_\alpha(L)$ the family $[K \mid K$ is a submaximal sublattice of L of the order $\alpha]$ and we define

$\Phi_\alpha(L) = \bigcap_{K \in \mathcal{K}_\alpha} K_{(L)}$. Evidently, $\Phi(L) = \Phi_1(L) = \Phi^1(L)$.

We shall call $\Phi_\alpha(L)$ (resp. $\Phi^\alpha(L)$) the iterated Frattini sublattice of the order α and of the type $\Phi_\alpha(L)$ (resp. $\Phi^\alpha(L)$). The sublattice $\Phi^n(L), n \in N$, has been defined in [3].

Theorem 7. For any lattices L_1, L_2 and any ordinal number α

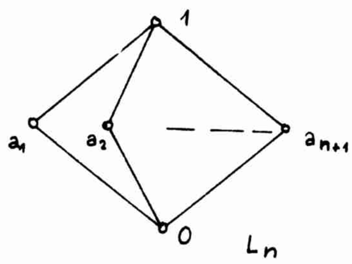
$$\Phi^\alpha(L_1 + L_2) = \Phi^\alpha(L_1) + \Phi^\alpha(L_2) ,$$

$$\Phi_\alpha(L_1 + L_2) = \Phi_\alpha(L_1) + \Phi_\alpha(L_2) .$$

Remark. For any positive integer n there exist lattices L, M such that $\Phi^n(L) \neq \emptyset$, $\Phi^{n+1}(L) = \emptyset$ and $\Phi_n(M) \neq \emptyset$, $\Phi_{n+1}(M) = \emptyset$.

In fact, it is sufficient to take $L = (L')_n$ where L' is the chain with $Card(L') = 2$ and $M = L_n$ is the lattice of Fig.2.

Fig.2



According to Fig.2, it is possible that there exist m, n positive integer such that $\Phi_m(L) = \Phi_{m+1}(L) \neq \emptyset$ and $\Phi_n(L) = \emptyset$.

However, it is not true in the case of the iterated Frattini sublattice of the type $\Phi^{\alpha}(L)$, as it can be deduced from the following consideration:

If $\beta < \alpha$ and $\Phi^{\beta}(L) = \Phi^{\alpha}(L)$, then $\Phi^{\beta}(L)$ contains no maximal sublattice and therefore $\Phi^{\beta}(L) = \Phi^{\gamma}(L)$ for all $\gamma > \beta$.

In [3], the problem

"Does the sequence $L \supseteq \Phi(L) \supseteq \Phi^{2}(L) \supseteq \dots$ always terminate?" is formulated.

Consider first that if L is a set, then the index set I of ordinal numbers such that $L \supsetneqq A_1 \supsetneqq \dots \supsetneqq A_\iota \supsetneqq \cdots$, $\iota \in I$ satisfies $Card(I) \leq Card(L)$.

It is obvious that there exists an ordinal number α such that $\Phi^{\alpha}(L) = \Phi^{\gamma}(L)$ for all $\gamma > \alpha$. But it is not certainly true that there always exists an ordinal number α such that $\Phi^{\alpha}(L) = \emptyset$. Indeed, let us observe the lattice K of Fig.3 (which has no maximal sublattice) or the L-sum of this lattice K where L is the lattice of Fig.4 and the element x is replaced by K.

Fig.3

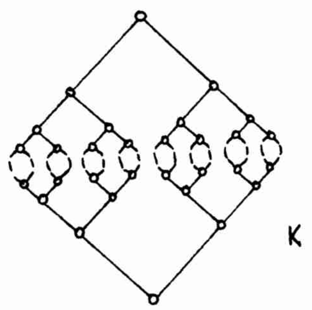
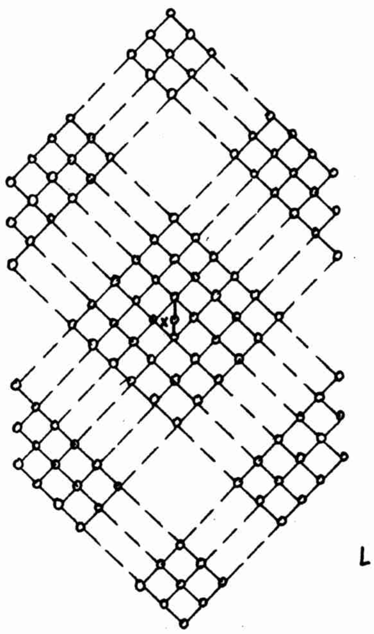


Fig. 4 (cf. [3], for $Card(L) < \aleph_0$)



Evidently, for the second construction the following claim is satisfied: For any ordinal number α there exists a lattice H such that: if $\alpha > \beta > \gamma$, then $\Phi^{\beta}(H) \subsetneq \Phi^{\gamma}(H)$, but for all ordinal numbers δ $\Phi^{\delta}(H) \neq \emptyset$.

A similar assertion holds for the iterated Frattini sublattices of the type $\Phi_{\alpha}(L)$, i.e.: For an arbitrary lattice L there exists an ordinal number α such that $\Phi_{\alpha}(L) = \Phi_{\beta}(L)$ for all $\beta > \alpha$.
Suppose it is not true, then there are ordinal numbers α, β such that $Card(L) = \aleph_{\delta}$, $\aleph_{\delta+1} \leq \alpha < \beta$ and

$\Phi_\alpha(L) \neq \Phi_\beta(L)$. Then there necessarily exists a submaximal sublattice K of L of the order β such that for each submaximal sublattice K' of the order $\omega \leq \alpha$ the inclusion $K \subseteq K'$ implies $K \subsetneqq K'$.

Let us have the sequence of submaximal sublattices K_ι of L of the order ι with $K_\beta = K$:

$L = K_0 \supseteq K_1 \supseteq \dots \supseteq K_\iota \supseteq \dots$.

If there exist two ordinal numbers ξ_1, ξ_2 with $\xi_1 < \xi_2$ and $K_{\xi_1} = K_{\xi_2}$, then $K_{\xi_1} = K_\xi$ for all $\xi > \xi_1$, therefore it is $K_{\delta_1} \supsetneqq K_{\delta_2}$ for all $\delta_1, \delta_2, \delta_1 < \delta_2 \leq \alpha$ and since $Card(L) = \aleph_\delta < \aleph_{\delta+1} \leq \alpha$ it gives a contradiction by the above remark.

<u>Summary</u>. Let L be a lattice with $Card(L) = \aleph_\delta$. Then there exist ordinal numbers α_1, α_2 such that α_1 , $\alpha_2 \leq \aleph_{\delta+1}$ and for any $\beta > \alpha_1$ $\Phi^{\alpha_1}(L) = \Phi^\beta(L)$, for any $\gamma > \alpha_2$ $\Phi_{\alpha_2}(L) = \Phi_\gamma(L)$.

6. <u>The lattice of all sublattices of a lattice</u>

In this chapter we shall assume that L is a nonempty lattice.

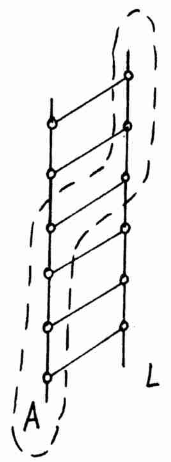
The lattice of all sublattices of L is denoted by $\mathcal{SL}(L)$, its lattice operations are denoted as follows: $A \cup_S B = \{A, B\}_L$, $A \cap_S B = A \cap B$.

$\mathcal{SL}(L)$ is a complete lattice with the least element $\emptyset$ and the greatest element L . Each sublattice A such that $Card(A) = 1$ is an atom of $\mathcal{SL}(L)$ and to every atom of $\mathcal{S}\ell(L)$ there corresponds a sublattice which consists of one element. A similar relation is between the maximal sublattices and the dual atoms of $\mathcal{S}\ell(L)$.

In this section we shall study the Frattini sublattice of L as an element of $\mathcal{S}\ell(L)$.

Evidently, if $\mathcal{S}\ell(L)$ is complemented, then $\Phi(L)$ is empty, but the converse does not hold, as it can be seen from Fig. 5: There exists no complement to the marked sublattice A in $\mathcal{S}\ell(L)$ though $\Phi(L)$ is empty.

Fig. 5



If K is a complete latfice, let us denote $rad(K) = \bigcap_{m \prec 1} m$ (cf.[4]) (if there exists no element $m \in K$ such that $m \prec 1$, we put $rad(K) = 1$). Obviously, $rad(\mathcal{S}\ell(L)) = \Phi(L)$.

We shall call an element b of a lattice M with the greatest element 1 small if $k \cup b \neq 1$ for all k of M, $k \neq 1$. It is immediate that if $h \leq k$ and k is small, then h is also small.

Theorem 8. Let L be a lattice. Then

$\Phi(L) = rad(\mathcal{SL}(L)) = \bigcup_s [A \mid A \text{ is small in } \mathcal{SL}(L)]$.

Proof. Clearly, if A is small in $\mathcal{SL}(L)$, then $A \subseteq \Phi(L)$. Let $B \supseteq A$ for all A small in $\mathcal{SL}(L)$, i.e., $\{h\}_L \subseteq B$ for all $\{h\}_L$ small in $\mathcal{SL}(L)$. As $\Phi(L) = [h \mid \{h\}_L$ is small in $\mathcal{SL}(L)]$, $\Phi(L)$ is contained in B.

$\Phi(L)$ is not necessarily small in $\mathcal{SL}(L)$ as it can be seen from Fig. 3.

Corollary 1. Let L be a latice.

If A is a sublattice of L and A is small in $\mathcal{SL}(L)$, then $A \subseteq \Phi(L)$.

If moreover $rad(\mathcal{SL}(L))$ is small in $\mathcal{SL}(L)$, then A is small in $\mathcal{SL}(L)$ iff A is a sublattice of $\Phi(L)$.

Corollary 2. The following conditions are equivalent:

1) $\mathcal{SL}(L)$ contains a small element different from $\emptyset$,

2) $rad(\mathcal{SL}(L)) \neq \emptyset$,

3) $\Phi(L) \neq \emptyset$.

7. The intersection of all maximal ideals of a lattice

Let L be a lattice, let us denote $\Phi I(L) = L$ if there exists no maximal ideal of L and $\Phi I(L) = \bigcap M$

otherwise, where M are maximal ideals of L.

In this chapter we shall compare the sublattices $\Phi(L)$ and $\Phi I(L)$. $\Phi I(L)$ has completely different properties than $\Phi(L)$, as it can be seen from the comparison of the corresponding assertions.

Lemma 3. Let L_1, L_2 be lattices, L_2 being nonempty. Then

$$\Phi I(L_1 + L_2) = L_1 + \Phi I(L_2) .$$

Corollary. 1) If L is nontrivially decomposable in an ordinal sum, then $\Phi I(L)$ is nonempty.

2) If L is a chain, then $\Phi I(L)$ is empty iff $\mathrm{Card}(L) \leq 1$.

3) If L is a lattice with $\mathrm{Card}(L) > 1$ such that every descending chain of reducible elements of L is finite, then $\Phi I(L)$ is nonempty.

Remark. For every lattice L there exists a lattice K such that $\Phi I(K) = L$. We can take, e.g., $K = L + L'$ where L' is a singleton.

Theorem 9. Let L_1, L_2 be lattices. Then

$$\Phi I(L_1 \times L_2) = \Phi I(L_1) \times \Phi I(L_2)$$

Proof. It is sufficient to realize that

a) I is an ideal of $L_1 \times L_2$ iff $I = I_1 \times I_2$ where I_1 is an ideal of L_1, I_2 is an ideal of L_2, $I_1 = [x \in L_1 \mid \exists y \in L_2$ such that $(x, y) \in I]$, $I_2 = [y \in L_2 \mid \exists x \in L_1$ such that $(x, y) \in I]$;

b) I is a maximal ideal of $L_1 \times L_2$ iff $I = I_1 \times I_2$ where either $I_1 = L_1$ and I_2 is a maximal ideal of L_2

or I_1 is a maximal ideal of L_1 and $I_2 = L_2$.

<u>Theorem 10</u>. Let L be a lattice with 1 . Then $\Phi I(L)$ is empty iff for all $h \in L$, $h \neq 1$ there exists an element $k \in L$, $k \neq 1$ such that $h \cup k = 1$.

<u>Proof</u>. Let $(a_1, a_2, \dots]$ denote the ideal generated by the set $[a_1, a_2, \dots]$. Let $h \in L$, $h \neq 1$, $h \notin \Phi I(L)$. Hence, there exists a maximal ideal I in L such that $(I, h] = 1$. Then there exists an element $k \in I$ such that $k \cup h = 1$, $k \neq 1$, because of $I \neq L$.
Let $h \neq 1$, $k \neq 1$, $h \cup k = 1$. Then $h \notin (k]$ and $((k], h] = L$. By Zorn's lemma, there exists a maximal ideal I_0 such that $h \notin I_0$ and $(k] \subseteq I_0$. I_0 is even a maximal ideal of L , hence $h \notin \Phi I(L)$.

The proof of the following lemma is immediate:

<u>Lemma 4</u>. Let $Card(L) > 1$, L be a lattice satisfying A.C.C. Then I is a maximal ideal of L iff $I = (a]$ for some dual atom a .

<u>Corollary</u>. Let L be a lattice satisfying A.C.C. and $Card(L) > 1$. Then $\Phi I(L) = [h \mid h$ is small in $L]$.

If in addition L is a complete lattice, then

$\Phi I(L) = [h \mid h$ is small in $L] = (rad(L)]$.

<u>Proof</u>. If h is small in L then $h \cup a < 1$ for all dual atoms, i.e., $h \leq a$ and so $h \in \Phi I(L)$.
If $k \in \Phi I(L)$, then $k \leq a$ for all dual atoms, i.e., k is small.

If L is complete, then $\Phi I(L) = \bigcap_{a \prec 1} (a] = (rad(L)]$.

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Matematicko-fyzikální fakulta

Karlova universita

Sokolovská 83

Československo

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