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Kontakt/Contact

[Digizeitschriften e.V.](#)
SUB Göttingen
Platz der Göttinger Sieben 1
37073 Göttingen

✉ info@digizeitschriften.de

ON SEQUENCES OF CONTRACTIVE-LIKE MAPPINGS AND FIXED POINTS

C.M. LEE, Milwaukee

Let (E, d) be a complete metric space, and n a positive integer. Recently, Kečkić ([2]) investigates sequences of mappings $f_m: E^n \rightarrow E$ satisfying the contractive-like condition:

$$(1) \quad d(f_m(u_1, u_2, \dots, u_n), f_{m+1}(u_2, u_3, \dots, u_{n+1})) \\ \leq \sum_{i=1}^n q_i d(u_i, u_{i+1}) \text{ for all } u_1, u_2, \dots, u_{n+1} \in E,$$

where q_1, q_2, \dots, q_n are non-negative constants with $\sum_{i=1}^n q_i < 1$. On the other hand, in unifying both Banach's contraction principle and Kannan's fixed point theorem, Reich ([4]) considers a single self-mapping f of E satisfying the condition:

$$(2) \quad d(f(x), f(y)) \leq ad(x, f(x)) + bd(y, f(y)) + cd(x, y) \\ \text{for all } x, y \in E,$$

where a, b, c are non-negative constants with $a + b + c < 1$. The purpose of this note is to consider a combined

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condition of (1) and (2), and obtain a result which contains both Kečkić's and Reich's results as particular cases. In fact, we prove

Theorem. Let (E, d) be a complete metric space, k a positive integer, f_m a mapping from E^k to E for $m = 1, 2, 3, \dots$. Suppose that

$$(3) \quad d(f_m(\mu_1, \mu_2, \dots, \mu_k), f_{m+1}(\mu_2, \mu_3, \dots, \mu_{k+1})) \\ \leq \sum_{i=1}^k \{ \alpha_i d(\mu_i, f_m(\mu_1, \mu_2, \dots, \mu_k)) + \beta_i d(\mu_{i+1}, f_{m+1}(\mu_2, \mu_3, \dots, \mu_{k+1})) \\ + \gamma_i d(\mu_i, \mu_{i+1}) \} + \delta_m \quad \text{for } m = 1, 2, 3, \dots,$$

and for $\mu_1, \mu_2, \dots, \mu_{k+1} \in E$,

where $\alpha_i, \beta_i, \gamma_i, \delta_m$ are non-negative constants and

$$\sum_{i=1}^k [(k-i+1)(\alpha_i + \beta_i) + \gamma_i] < 1, \quad \sum_{m=1}^{+\infty} \delta_m < +\infty.$$

Define $x_{m+k} = f_m(x_m, x_{m+1}, \dots, x_{m+k-1})$ for $m = 1, 2, 3, \dots$,

where x_1, x_2, \dots, x_k are chosen arbitrarily from E . Then

(i) $\{x_{n+k}\}$ converges in E ;

(ii) if $f_m(u, u, \dots, u) \rightarrow f(u, u, \dots, u)$ as $m \rightarrow +\infty$

for each $u \in E$, then $f(u, u, \dots, u) = u$ has x as

a solution, where $x = \lim_{m \rightarrow +\infty} x_{m+k}$; if, furthermore,

$$\alpha_i = \alpha_{k-i+1}, \quad \beta_i = \beta_{k-i+1} \quad \text{for } i = 1, 2, 3, \dots, \left[\frac{k}{2}\right],$$

then x is unique to satisfy $f(u, u, \dots, u) = u$;

(iii) if f and α_i, β_i and x are as in (ii), and $f_m(y_m, y_m, \dots, y_m) = y_m$ for $m=1, 2, 3, \dots$, then $y_m \rightarrow x$, the unique solution of $f(u, u, \dots, u) = u$.

Proof. To show that $\{x_{m+k}\}$ converges in E , it suffices to show that $\{x_{m+k}\}$ is a Cauchy sequence in E since E is complete. To this end, let $D_m = d(x_m, x_{m+1})$ for $m=1, 2, 3, \dots$. Then $\therefore m=1, 2, 3, \dots$, using (3) and the triangle inequality, one has

$$\begin{aligned} D_{m+k} &\leq \sum_{i=1}^k \{ \alpha_i d(x_{m+i-1}, x_{m+k}) + \beta_i d(x_{m+i}, x_{m+k+1}) + \gamma_i d(x_{m+i-1}, x_{m+i}) \} + \sigma_m \\ &\leq \sum_{i=1}^k \{ \alpha_i [D_{m+i-1} + D_{m+i} + \dots + D_{m+k-1}] + \beta_i [D_{m+i} + D_{m+i+1} + \dots + D_{m+k}] + \gamma_i D_{m+i-1} \} + \sigma_m . \end{aligned}$$

Hence by simple calculations one has

$$\sum_{m=1}^n D_{m+k} \leq b \sum_{i=1}^k D_i + b \sum_{n=1}^m D_{m+k} + \sum_{n=1}^m \sigma_n ,$$

where $b = \sum_{i=1}^k [(\ell-i+1)(\alpha_i + \beta_i) + \gamma_i] < 1$. Therefore

$$\sum_{n=1}^m D_{m+k} \leq \frac{1}{1-b} [b \sum_{i=1}^k D_i + \sum_{n=1}^m \sigma_n] \leq \frac{1}{1-b} [b \sum_{i=1}^k D_i + \sum_{n=1}^{+\infty} \sigma_n] ,$$

of which the right hand side is independent of m . Thus,

we conclude that $\sum_{n=1}^{+\infty} D_{m+k}$ is a convergent series.

Now, for $m > n$, $d(x_{m+k}, x_{n+k}) \leq \sum_{i=n}^m D_{i+k} \rightarrow 0$ as

$m \rightarrow +\infty$, proving that $\{x_{m+k}\}$ is a Cauchy sequence.

To show that $f(x, x, \dots, x) = x$, where

$x = \lim_{m \rightarrow +\infty} x_{m+k}$, let us denote

$$x_{m+k}^i = f_{m+i}(x_{m+i}, x_{m+i+1}, \dots, x_{m+k-1}, x, x, \dots, x)$$

for $m = 1, 2, 3, \dots$, $i = 0, 1, 2, \dots, k$. Note that, in

particular, $x_{m+k}^0 = f_m(x_m, x_{m+1}, \dots, x_{m+k-1}) = x_{m+k}$,

$x_{m+k}^k = f_{m+k}(x, x, \dots, x)$. Then for $m = 1, 2, 3, \dots$, we

have

$$\begin{aligned} (A) \quad & d(x, f(x, x, \dots, x)) \\ & \leq d(x, x_{m+k}^0) + d(x_{m+k}^0, x_{m+k}^1) + \dots + d(x_{m+k}^{k-1}, x_{m+k}^k) \\ & \quad + d(f_{m+k}(x, x, \dots, x), f(x, x, \dots, x)) \end{aligned}$$

by the triangle inequality. Using (3) and then the triangle inequality, if we denote

$$\Delta_m^j = \Delta_m^j = d(x_{m+k}^j, x_{m+k}^{j+1}) \quad \text{for } j = 0, 1, 2, \dots, k-1,$$

then we obtain

$$\begin{aligned} (B) \quad \Delta^0 & \leq \frac{1}{1-\beta} \left[\sum_{l=m}^{m+k-1} \beta_l + d(x, x_{m+k}^0) + d(x, x_{m+k-1}) + \sigma_m \right] \\ \Delta^j & \leq \frac{1}{1-\beta} \left[\sum_{l=m}^{m+k-1} \beta_l + d(x, x_{m+k}^0) + d(x, x_{m+k-1}) + \sigma_{m+j} \right. \\ & \quad \left. + (\Delta^0 + \Delta^1 + \dots + \Delta^{j-1}) \right] \end{aligned}$$

for $j = 1, 2, 3, \dots, k-1$,

where $\beta = \sum_{i=1}^k \beta_i < 1$.

Given $\varepsilon > 0$, choose m so large that

$$\sum_{l=m}^{m+k-1} D_l < \frac{\varepsilon_1}{3}, d(f_{m+k}(x, x, \dots, x), f(x, x, \dots, x)) < \frac{\varepsilon_1}{3},$$

$$d(x, x_{m+k-1}) < \frac{\varepsilon_1}{6}, d_m^* < \frac{\varepsilon_1}{3} \text{ for all } m \geq m, \text{ where}$$

$$\varepsilon_1 = \frac{(1-\beta)^{k-1} \varepsilon}{A} \text{ with } A = 2 \sum_{j=0}^{k-1} \left(1 + \frac{j(j+1)}{2}\right),$$

noting that such an m can be chosen since

$$\sum_{l=1}^{+\infty} D_l < +\infty, f_m \rightarrow f, x_{m+k} \rightarrow x \text{ and } \sum_{m=1}^{+\infty} d_m^* < +\infty.$$

Then from (B) we have

$$\Delta^j < \frac{\varepsilon}{A} \left(1 + \frac{j(j+1)}{2}\right) (1-\beta)^{k-(j+1)} \leq \frac{\varepsilon}{A} \left(1 + \frac{j(j+1)}{2}\right)$$

for $j = 0, 1, 2, \dots, k-1$,

and hence we have from (A) that

$$\begin{aligned} d(x, f(x, x, \dots, x)) &< \frac{\varepsilon_1}{6} + \frac{\varepsilon_1}{3} + \Delta^0 + \Delta^1 + \dots + \Delta^{k-1} \\ &< \frac{\varepsilon_1}{2} + \frac{\varepsilon}{A} \sum_{j=0}^{k-1} \left(1 + \frac{j(j+1)}{2}\right) = \frac{\varepsilon_1}{2} + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

As ε is arbitrary, we conclude that $f(x, x, \dots, x) = x$.

To prove the uniqueness, suppose that

$$y = f(y, y, \dots, y). \text{ For convenience, let } \bar{x}_m^i = f_{m+i}(x, x, \dots, x, y_1, y_2, \dots, y_i) \text{ for } i = 0, 1, 2, \dots, k,$$

$m = 1, 2, 3, \dots$, where $y_1 = y_2 = \dots = y_i = y$, noting

$$\text{that } \bar{x}_m^0 = f_m(x, x, \dots, x), \bar{x}_m^k = f_{m+k}(y, y, \dots, y).$$

$$\text{Also, let } \bar{\Delta}^i = \bar{\Delta}_m^i = d(\bar{x}_m^i, \bar{x}_m^{i+1}) \text{ for } i =$$

$= 0, 1, 2, \dots, k-1$. Then by the triangle inequality,

$$\begin{aligned}
(C) \quad d(x, y) &= d(f(x, x, \dots, x), f(y, y, \dots, y)) \\
&\leq d(f(x, x, \dots, x), f_m(x, x, \dots, x)) + d(f(y, y, \dots, y), \\
&\quad f_{m+k}(y, y, \dots, y)) + \bar{\Delta}^0 + \bar{\Delta}^1 + \bar{\Delta}^2 + \dots + \bar{\Delta}^{k-1}.
\end{aligned}$$

Now, by (3) and then by the triangle inequality again, we have

$$\begin{aligned}
\bar{\Delta}^0 &\leq \sum_{i=1}^k \alpha_i d(x, \bar{x}_m^0) + \sum_{i=1}^{k-1} \beta_i [d(x, \bar{x}_m^0) + \bar{\Delta}^0] \\
&+ \beta_k [d(y, \bar{x}_m^k) + \bar{\Delta}^1 + \bar{\Delta}^2 + \dots + \bar{\Delta}^{k-1}] + \gamma_k d(x, y) + \sigma_m'; \\
&+ \sum_{i=1}^{k-2} \beta_i [d(x, \bar{x}_m^0) + \bar{\Delta}^0 + \bar{\Delta}^1] + (\beta_k + \beta_{k-1}) [d(y, \bar{x}_m^k) + \bar{\Delta}^2 + \bar{\Delta}^3 + \dots + \bar{\Delta}^{k-1}] \\
&+ \sum_{i=1}^{k-2} \beta_i [d(x, \bar{x}_m^0) + \bar{\Delta}^0 + \bar{\Delta}^1] + (\beta_k + \beta_{k-1}) [d(y, \bar{x}_m^k) + \bar{\Delta}^2 + \bar{\Delta}^3 + \dots + \bar{\Delta}^{k-1}] \\
&+ \gamma_{k-1} d(x, y) + \sigma_{m+1}'; \\
\bar{\Delta}^{k-1} &\leq \alpha_1 [d(x, \bar{x}_m^0) + \bar{\Delta}^0 + \bar{\Delta}^1 + \dots + \bar{\Delta}^{k-2}] \\
&+ \sum_{i=2}^k \alpha_i [d(y, \bar{x}_m^k) + \bar{\Delta}^{k-1}] + \sum_{i=1}^k \beta_i d(y, \bar{x}_m^k) \\
&+ \gamma_1 d(x, y) + \sigma_{m+k-1}'.
\end{aligned}$$

Hence, using the condition $\alpha_i = \alpha_{k-i+1}$, $\beta_i = \beta_{k-i+1}$

for $i = 1, 2, \dots, [\frac{k}{2}]$, we have

$$\begin{aligned}
\bar{\Delta}^0 + \bar{\Delta}^1 + \dots + \bar{\Delta}^{k-1} &\leq \frac{1}{1 - (t - (\alpha + \beta))} \{ t [d(x, \bar{x}_m^0) + \\
&+ d(y, \bar{x}_m^k)] + \gamma d(x, y) + \sum_{i=1}^k \sigma_{m+i-1}' \}, \quad \text{where} \\
\alpha &= \sum_{i=1}^k \alpha_i, \beta = \sum_{i=1}^k \beta_i, \gamma = \sum_{i=1}^k \gamma_i \quad \text{and} \quad t = \sum_{i=1}^k (k-i+1)(\alpha_i + \beta_i).
\end{aligned}$$

Hence from (C) we have

$$\left(1 - \frac{\gamma}{1 - (t - (\alpha + \beta))}\right) d(x, y) \leq A(m),$$

where $A(m)$ is an expression such that $A(m) \rightarrow 0$ as $m \rightarrow +\infty$.

Noting that $1 - \frac{\gamma}{1 - (t - (\alpha + \beta))} > 0$ by the condition

that $\sum_{i=1}^k [(\kappa - i + 1)(\alpha_i + \beta_i) + \gamma_i] < 1$, we conclude

that $d(x, y) = 0$, so that $x = y$, proving the uniqueness in part (ii).

With a slight modification in the proof of uniqueness, one proves easily that $d(y_n, x) \rightarrow 0$ as $n \rightarrow +\infty$, so that $y_n \rightarrow x$ as $n \rightarrow \infty$ follows.

Remarks. (I) Reich's result in [4] is obtained from our theorem (part (ii)) by taking $\kappa = 1$, $f = f_n$ for $n = 1, 2, 3, \dots$.

(II) If $\alpha_i = \beta_i = 0$ for $i = 1, 2, 3, \dots, \kappa$, then one sees that $f_{n_i}(u, u, \dots, u) \rightarrow f(u, u, \dots, u)$ uniformly, so that our result (part (i) and (ii)) contains that of Kečkić's in [2].

(III) Part (iii) gives some kind of sufficient conditions for a sequence of fixed points of functions to converge to the fixed point of the convergent function. For other kinds of sufficient conditions, we refer to Bonsall [1], Nadler [3], and also Singh and Russell [5].

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University of Wisconsin-Milwaukee
Milwaukee, Wisconsin, U.S.A.

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