Commentationes Mathematicae Universitatis Carolinae

13,3 (1972)

* -BIREGULAR RINGS

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Introduction. Regular rings were first defined by von Neumann [1] and used in connection with continuous geometries, there being an isomorphism between a continuous geometry and all principal left ideals of some regular ring. The theory was later expanded by introducing the notion of a * -regular ring, and biregular rings were developed as a two-sided analogue to regularity. It is the purpose of this paper to develop a two-sided analogue to * -regularity, and to produce an isomorphism theorem analogous to the above.

1. Regular, * -regular and biregular rings.

1.1. Definition. An associative ring R with a unit is regular if $axa = a$ is solvable in R for all $a \in R$.

1.2. Definition. A regular rings is * -regular if there exists an involutory anti-automorphism $a \rightarrow a^*$ of the ring onto itself, such that $aa^* = 0$ if and only if $a = 0$.

If R is * -regular an element $a \in R$ for which $a = a^*$ is called self-conjugate. Self-conjugate idempotents are called projections.

AMS, Primary: 16A30 Ref. Ž. 2.723.2

We have the following properties (proved in [3]).

1.3. **Theorem.** If R is an associative ring with unit, then

i) R is regular if and only if every principal left ideal of R is generated by a unique idempotent.

ii) R is $*$-regular if and only if every principal left ideal of R is generated by a unique projection.

As a two-sided analogue to regularity we have the following.

1.4. **Definition.** A ring is said to be **biregular** if every principal ideal is generated by a central idempotent.

2. $*$-Biregular rings.

In view of Theorem 1.3 we would expect that the defining criterion for a two-sided analogue to $*$-regularity would be that every principal two-sided ideal of such a ring be generated by a unique central projection. Our two-sided analogue to a $*$-regular ring will be defined as follows.

2.1. **Definition.** A ring is defined to be $*$**-biregular** if it is both biregular and $*$-regular.

2.2. **Theorem.** Every principal ideal in a $*$-biregular ring R is generated by a uniquely defined central projection.

Proof. Let I be a principal two-sided ideal in R. Then, since R is biregular, I is generated by a central idempotent e. We see immediately that $(e^*)^2 = e^*$, and that $(ee^*)^2 = ee^*$. Therefore $(1-ee^*)ee^* = 0$, and so $(1-ee^*)ee^*(1-ee^*)^* = [(1-ee^*)e].[(1-ee^*)e]^* = 0$, which implies that $e = ee^*e = ee^* = e^*e$. e^* is central since, if x is an arbitrary member of R , $e^*x = (x^*e)^* = (ex^*)^* = xe^*$. Obviously now, $(e)R = (ee^*)R = I$, and ee^* is a central projection.

If $I = pR$, where p is a central projection, then $p = ee^*x$ and $ee^* = py$ for some $x, y \in R$. Then $p = ee^*p = pee^* = ee^*$, and so we have uniqueness.

We can give a further description of the above projection ee^* by means of the following.

2.3. Theorem. If R is a $*$ -biregular ring and I_a is a principal ideal of R generated by a , then the unique central projection which generates I_a is the least central element such that $ad = a$.

Proof. I_a is the set of all finite sums $\sum_i x_i a y_i$, where $x_i, y_i \in R$, $i = 1, 2, \dots$. Also, $I_a = eR$ where e is a central idempotent, and by the previous theorem, $I_a = ee^*R$, where ee^* is a central projection. Then $a = ee^*z$ for some $z \in R$ and therefore $aee^* = ee^*zee^* = (ee^*)^2 z = ee^*z = a$. Thus $a(ee^*) = a$ and ee^* is central.

Now let d be a central element such that $ad = a$. Then $ee^* = \sum_i x_i a y_i = \sum_i x_i a d y_i = d \sum_i x_i a y_i = dee^*$. Therefore we have $ee^*R = dee^*R \subseteq dR$, i.e. $ee^* \leq d$.

The center of a biregular ring is biregular ([4], Theorem 4). We also prove the following result.

2.4. Theorem. The center of a $*$ -regular ring is $*$ -regular.

Proof. It is well known that the center of a regular ring is regular, and therefore we need only show that if a is in the center, then so is a^*. Let $a \in Z$, where Z is the center, and let x be an arbitrary element of R. Then $a^*x = (x^*a)^* = (ax^*)^* = xa^*$, i.e. a^* is central.

Therefore the center of a $*$-biregular ring is both biregular and $*$-regular, and we get

2.5. Theorem. The center of a $*$-biregular ring is $*$-biregular.

A $*$-regular ring is said to be complete if the lattice of its projections is complete, and Kaplansky [5] has shown that if a $*$-regular ring is complete then its projections form a continuous geometry. If the ring is commutative, then the principal one-sided ideals are in fact principal two-sided ideals. Therefore, if the center of a $*$-biregular ring is complete, the lattice of its principal ideals form a continuous geometry.

Morrison ([4], Theorem 5) has shown that there is an isomorphism between the principal ideals of the center of a biregular ring and the principal ideals of the ring itself. We therefore get the following.

2.6. Theorem. The lattice of the principal ideals of a $*$-biregular ring R, whose center is complete, is a continuous geometry, i.e. the central projections of a $*$-biregular ring form a continuous geometry.

This, of course, is the two-sided analogue to Kaplansky's result.

The following theorem is one of the main results of von Neumann [2].

2.7. Theorem. A complemented modular lattice admitting a homogeneous basis of rank ≥ 4 has orthocomplements if and only if it is isomorphic to the lattice of principal left ideals of some $*$-regular ring.

In a two-sided analogue to this theorem we would want to replace "the lattice of principal left ideals of some $*$-regular ring" by " the lattice of principal ideals of some $*$-biregular ring".

Now, a $*$-biregular ring is biregular, and the lattice of principal ideals of a biregular ring is a distributive, relatively complemented lattice (Andrunakievich [6]). If the ring contains a unit (which is the case for a $*$-biregular ring, since a $*$-biregular ring is regular and a regular ring has a unit) then this lattice is a Boolean algebra. A Boolean algebra is certainly orthocomplemented and so we seek to prove the following

2.8. Theorem. A Boolean algebra $\mathfrak{B}$ is isomorphic to the lattice of principal ideals of some $*$-biregular ring.

Proof. Every Boolean algebra $\mathfrak{B}$ is isomorphic to the lattice of principal ideals of some Boolean ring $\mathfrak{R}$ (Birkhoff,[7], p.155). Trivially, a Boolean ring is commutative, regular and biregular. The commutativity gives us that the identity mapping is an anti-automorphism $a \rightarrow a^*$ of $\mathfrak{R}$ onto itself. Also $aa^* = 0$ implies $a = a^2 = aa^* = 0$, since every element of a Boolean ring is an idempotent. Therefore $\mathfrak{R}$ is $*$-regular and biregular, and hence is $*$-biregular.

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(Oblatum 15.3. 1972)