Commentationes Mathematicae Universitatis Carolinae

13,3 (1972)

TWO NOTES ON RADICALS OF ABELIAN GROUPS

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Introduction. Since its introduction for rings and algebras by Kurosh [15] and Amitsur [1], the general theory of radicals has been extended to other algebraic structures, notably groups and modules. In this paper we shall work in the class of abelian groups, which, as a setting for radical theory, is related to each of the foregoing, being the class of modules over a ring, a universal class of groups and, when regarded as the class of zerorings, a universal class of rings as well. There is another connection with rings: if $\mathcal{R}$ is a radical class of abelian groups, the rings whose additive groups belong to $\mathcal{R}$ form a radical class. Such classes (A -radical classes) were introduced in [13] and a further application is contained in [14].

We begin by recalling the basic terminology concerning radical classes (known elsewhere as torsion classes) of abelian groups. The standard references are [16],[5],[6],[17]. A radical class is a non-void homomorphically closed class $\mathcal{R}$ such that every abelian group G has a largest subgroup $\mathcal{R}(G)$ belonging to $\mathcal{R}$ and $\mathcal{R}(G/\mathcal{R}(G)) = 0$ for

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AMS, Primary: 20K99, 18E40 Ref. Ž. 2.722.1

each G . A non-empty class $\mathcal{R}$ is a radical class if and only if it is closed under formation of homomorphic images, extensions and direct sums. If $\mathcal{R}$ is a radical class, a group G with $\mathcal{R}(G) = 0$ is said to be $\mathcal{R}$ - <u>semi-simple</u>. Every non-empty class $\mathcal{C}$ is contained in a smallest radical class, namely

$$L(\mathcal{C}) = \{A \mid [C, B] = 0 \ \forall C \in \mathcal{C} \Rightarrow [A, B] = 0\}$$

called the <u>lower radical class</u> determined by $\mathcal{C}$. (Here $[A, B]$ is the group of homomorphisms from A to B .) There is also a largest radical class with respect to which all members of $\mathcal{C}$ are semi-simple: the <u>upper radical class</u> determined by $\mathcal{C}$,

$$\mathcal{U}(\mathcal{C}) = \{A \mid [A, C] = 0 \ \forall C \in \mathcal{C}\} .$$

If $\mathcal{C}$ has a single member C , we shall write $L(C)$, $\mathcal{U}(C)$ rather than $L(\{C\})$, $\mathcal{U}(\{C\})$.

The two sections of the paper are virtually disjoint. in § 1 we prove that for every radical class $\mathcal{R}$ and every abelian group G , $\mathcal{R}(G)$ is a pure subgroup and consider some implications. § 2 is devoted to an examination of the behaviour of rational groups as members of radical classes. It has been shown [5] that any radical class is the lower radical class determined by its torsion and torsion-free members and the behaviour of torsion groups in this context is completely known. Thus the investigation carried out in this section is the logical next step in any attempt at a classification of radical classes.

We adopt the usual convention of using the word "group" to mean "abelian group" and our notation for groups follows [10] with the following exceptions: $[A, B]$ is the group of homomorphisms from a group A to a group B; G_t (resp. G_p) is the maximum torsion (resp. p-primary) subgroup of G; $T(X)$ is the type of a rational group X, $T(x)$ the type of a group element x.

1. The purity of radicals.

Proposition 1.1. For every radical class $\mathcal{R}$ and every group G, $\mathcal{R}(G)$ is a pure subgroup.

Proof. Let p be a prime. If $Z(p) \notin \mathcal{R}$, then $\mathcal{R}(G)$ is p-divisible ([5], Lemma 5.1) and hence $\mathcal{R}(G) \cap p^m G = p^m \mathcal{R}(G)$ for each m. The latter equation is also valid when $Z(p) \in \mathcal{R}$, since then $G/\mathcal{R}(G)$ has no element of order p ([5], Lemma 2.1). //

Corollary 1.2. If A and B are torsion-free groups for which there exists an exact sequence

$$0 \to A \xrightarrow{f} B \to C \to 0$$

with C a torsion group, then any radical class which contains A also contains B.

Proof. Suppose a radical class $\mathcal{R}$ contains A but not B. Then $\mathcal{R}(B)$ is a proper pure subgroup of B containing $f(A)$ and hence $B/\mathcal{R}(B)$ is a non-zero torsion-free homomorphic image of C, which clearly is impossible. //

The last result enables us to exhibit another closure property for radical classes. Recall that two groups G and

H are <u>quasi-isomorphic</u> ($G \overset{\cdot}{\cong} H$) if they have subgroups G', H' respectively such that G/G' and H/H' are bounded and $G' \cong H'$.

<u>Theorem 1.3</u>. Radical classes are closed under quasi-isomorphisms.

<u>Proof</u>. Let $\mathcal{R}$ be a radical class and $H \overset{\cdot}{\cong} G \in \mathcal{R}$. Then G_t, $G/G_t \in \mathcal{R}$ ([5], Theorem 5.2) and it is easily seen that $H_t \overset{\cdot}{\cong} G_t$ and $H/H_t \overset{\cdot}{\cong} G/G_t$. By a result of Beaumont and Pierce [4], $H_p \cong G_p$ for almost all primes p, and for the others, $H_p \overset{\cdot}{\cong} G_p$. If $\mathcal{R}$ contains no non-zero p-groups, the latter condition implies $G_p = 0 = H_p$; if $\mathcal{R}$ contains just the divisible p-groups, then $H_p \cong G_p$ and if $\mathcal{R}$ contains all p-groups, it contains H_p. Since the class of p-groups in $\mathcal{R}$ must be one of those mentioned ([5],[16]) it follows that $H_t \in \mathcal{R}$. Another result of the same authors [3] implies the existence of an exact sequence

$$0 \to G/G_t \to H/H_t \to B \to 0$$

where B is bounded, whence it follows from Corollary 1.2 that $H/H_t \in \mathcal{R}$. Thus $H \in \mathcal{R}$ and the theorem is proved. //

A question suggested by Proposition 1.1 is: For which radical classes $\mathcal{R}$ is $\mathcal{R}(G)$ always a direct summand ?

<u>Proposition 1.4</u>. A non trivial radical class $\mathcal{R}$ is such that $\mathcal{R}(G)$ is a direct summand for every group G if and only if $\mathcal{R} \subseteq \mathcal{D}$, the class of divisible groups (and thus $\mathcal{R} = \mathcal{D}$ or the class of divisible groups of the form $\bigoplus G_p$, $p \in P$ for some set P of primes).

Proof. If $\mathcal{R} \not\subseteq \mathcal{D}$, then $Z(p) \in \mathcal{R}$ for some prime p, whence so are all p-groups ([5],[16]). As $\mathcal{R}$ is non-trivial, Z is $\mathcal{R}$-semi-simple (see Proposition 2.4 below) as is $\prod Z_n$, $n = 1, 2, \ldots$; $Z_n \cong Z$. By a theorem of Baer [2], Erdös [8] and Sąsiada (see [9] p. 190) there exists a non-split exact sequence

$$0 \to A \xrightarrow{f} B \to \prod Z_n \to 0$$

where A is a reduced unbounded p-group, and hence belongs to $\mathcal{R}$. It follows that $\mathcal{R}(B) = f(A)$ is not a direct summand. Thus $\mathcal{R} \subseteq \mathcal{D}$ if $\mathcal{R}(G)$ is always a direct summand. The converse is clear. (That $\mathcal{D}$ has no proper radical cubclasses which contain torsion-free groups follows from Corollary 2.3 below.) //

2. Rational groups and radical classes.

The following result is an immediate consequence of Proposition 1.1:

Proposition 2.1. Let X be a rational group and $\mathcal{R}$ a radical class. Then $\mathcal{R}(X) = X$ or 0. //

Proposition 2.2. Let X and Y be rational groups. Then $Y \in L(X)$ if and only if $T(X) \leq T(Y)$.

Proof. If $T(X) \leq T(Y)$ then $[X, Y] \neq 0$, so Y is not $L(X)$-semi-simple, and hence by Proposition 2.1, $Y \in L(X)$. Conversely, if $Y \in L(X)$, there is a non-zero homomorphism (necessarily an injection) from X to Y, so $T(X) \leq T(Y)$. //

Corollary 2.3. Any radical class $\mathcal{R}$ which contains torsion-free groups must contain Q. Hence $\mathcal{D} = L(Q)$

is the unique smallest radical class with torsion-free members.

<u>Proof</u>. If $\mathcal{R}$ contains a torsion-free group $G \neq 0$, it contains any rational homomorphic image X of G; hence $Q \in L(X) \subseteq \mathcal{R}$. The proof that $\mathcal{D} = L(Q)$ follows that of [5], Proposition 4.1. //

On the other hand any radical class which contains Z must contain all free groups and hence all groups. Thus we obtain

<u>Proposition 2.4</u>. $\mathcal{U}(Z)$, the class of groups without free direct summands, is the largest non-trivial radical class. //

<u>Definition 2.5</u>. An <u>r.t. radical class</u> is the lower radical class determined by a collection of torsion and rational groups.

<u>Definition 2.6</u>. The <u>type set</u> of a radical class $\mathcal{R}$ is the set of types in rational groups in $\mathcal{R}$.

We now proceed to classify the r.t. radical classes by their type sets and torsion members.

<u>Proposition 2.7</u>. A non-empty set Γ of types is the type set of a radical class if and only if it satisfies

$$(*) \qquad \gamma \in \Gamma, \quad \chi \geq \gamma \Rightarrow \chi \in \Gamma .$$

<u>Proof</u>. Let $\mathcal{R}$ be a radical class with type set Γ. If X, Y are rational groups with $T(X) \leq T(Y)$ and $X \in \mathcal{R}$, then by Proposition 2.2, $Y \in L(X) \subseteq \mathcal{R}$. Thus Γ satisfies $(*)$. Conversely, if Γ is a non-empty set of types satisfying $(*)$, let

$$\mathcal{R} = L(\{X \text{ rational} \mid T(X) \in \Gamma\}) .$$

If Y is rational and belongs to $\mathcal{R}$, then $[X, Y] \neq 0$ for some rational X with $T(X) \in \Gamma$, so $T(Y) \geq$ $\geq T(X)$ and hence $T(Y) \in \Gamma$. Thus Γ is the type set of $\mathcal{R}$. //

__Definition 2.8.__ Let Γ be a set of types satisfying $(*)$, P a set of primes such that every rational group X with $T(X) \in \Gamma$ is p-divisible for all $p \in P$. $\mathcal{R}[\Gamma, P]$ is the radical class

$$L(\{X \text{ rational} \mid T(X) \in \Gamma\} \cup \{Z(p) \mid p \in P\}).$$

__Theorem 2.9.__ A radical class is an r.t. radical class if and only if it has the form $\mathcal{R}[\Gamma, P]$. Such a representation is unique.

__Proof.__ If $\mathcal{R}$ is an r.t. radical class, then by [5], Theorem 2.6 we may assume

$$\mathcal{R} = L(\{X_i \mid i \in I\} \cup \{Z(p) \mid p \in P_1\} \cup \{Z(p^\infty) \mid p \in P_2\})$$

where P_1, P_2 are disjoint sets of primes and the X_i are rational groups. Let Γ be the type set of $\mathcal{R}$ and

$$P = \{p \in P_1 \mid X \text{ rational}, X \in \mathcal{R} \Rightarrow pX = X\}.$$

We shall prove that $\mathcal{R} = \mathcal{R}[\Gamma, P]$.

Since $L(\{X_i \mid i \in I\})$ contains all divisible groups (Corollary 2.3), $Z(p)$ (and hence all p-groups) for $p \in P_1 - P$ [5], we have

$$\{Z(p) \mid p \in P_1 - P\} \cup \{Z(p^\infty) \mid p \in P_2\} \subseteq L(\{X_i \mid i \in I\})$$

whence

$$\mathcal{R} \subseteq L(\{X_i \mid i \in I\}) \cup \{Z(p) \mid p \in P\}) \subseteq \mathcal{R}.$$

That $\mathcal{R} = \mathcal{R}[\Gamma, P]$ and the representation is unique can now readily be deduced from the characterization of the lower radical class given in the introduction. //

Example 2.10. A non-r.t. radical class: Let G be indecomposable, torsion-free with rank > 1 and homogeneous of type τ. Any rational homomorphic image X of G has type $\geq \tau$ and equality is impossible by a result of Baer (see [9] p.163). But then $[X, G] = 0$. It follows that $L(G)$ is not an r.t. radical class.

Some other examples are furnished by the following result.

Theorem 2.11. For no radical class $\mathcal{R}$ do there exist non-empty classes $\mathcal{A}$, $\mathcal{B}$ of rational groups such that $L(\mathcal{A}) = \mathcal{R} = \mathcal{U}(\mathcal{B})$.

Proof. Assume, on the contrary, that such classes $\mathcal{A}$, $\mathcal{B}$, $\mathcal{R}$ exist. By Corollary 2.3 and Proposition 2.4 we have $Q \in \mathcal{A}$ and $Z \in \mathcal{B}$. Let G be a torsion-free group of rank > 1 which is homogeneous of type $T(Z)$ and for which all proper torsion-free homomorphic images are divisible. (The existence of such a group is guaranteed by [7], Theorem 4.) $[X, G] = 0$ for each $X \in \mathcal{A}$, so G is $\mathcal{R}$-semi-simple, but $[G, Y] = 0$ for all $Y \in \mathcal{B}$, so G is $\mathcal{R}$-radical. Hence there is no such $\mathcal{R}$. //

We next consider r.t. radical classes with type sets Γ satisfying the additional condition

(**) $\sigma, \tau \in \Gamma \Rightarrow \sigma \cap \tau \in \Gamma$,

i.e. Γ is a dual ideal in the lattice of all types. If Γ

is such a set, then for every torsion-free group G, the set

$$G(\Gamma) = \{x \in G \mid T(x) \in \Gamma\}$$

is a pure subgroup, since if $x, y \in G(\Gamma)$ we have $T(x-y) \geq T(x) \cap T(y) \in \Gamma$ (0 is regarded as having a type greater than all others) and $T(nx) = T(x)$ for non-zero integers n. When Γ is the principal dual ideal generated by a type τ, $G(\Gamma)$ is the subgroup customarily called $G(\tau)$. Clearly $G(\Gamma) \subseteq \mathcal{R}[\Gamma, P](G)$ for every torsion-free group G. The following theorem gives some further information about these two subgroups.

__Theorem 2.12__. Let $\mathcal{R} = \mathcal{R}[\Gamma, P]$ be an r.t. radical class for which Γ is a dual ideal in the lattice of types and for torsion-free groups G let

$$G(\Gamma) = \{x \in G \mid T(x) \in \Gamma\} .$$

The following conditions are equivalent:

(i) The class $\mathcal{C}(\Gamma) = \{G$ torsion-free $\mid G(\Gamma) = G\}$ is closed under extensions.

(ii) $(G/G(\Gamma))(\Gamma) = 0$ for all torsion-free groups G.

(iii) $G(\Gamma) = \mathcal{R}(G)$ for all torsion-free groups G.

(iv) $\mathcal{R}$ is pure-hereditary (i.e. pure subgroups of groups in $\mathcal{R}$ belong to $\mathcal{R}$).

(v) Γ is the principal dual ideal generated by the type of a height sequence $(h_1, h_2, \ldots)$ where h_n takes only the values 0 and ∞.

__Proof__. (i) $\Longrightarrow$ (ii): For a torsion-free group G, let G' be the subgroup defined by the exact sequence

$$0 \to G(\Gamma) \to G' \to (G/G(\Gamma))(\Gamma) \to 0 .$$

Then by assumption $G' \in \mathcal{C}(\Gamma)$. If x is an element of G', its type in G is at least as great as that in G', so $G' \subseteq G(\Gamma)$, i.e. $(G/G(\Gamma))(\Gamma) = 0$.

(ii) $\Longrightarrow$ (iii): $G(\Gamma) \in \mathfrak{R}$ for any torsion-free G and $[X, G/G(\Gamma)] = 0$ for every rational group X with $T(X) \in \Gamma$. Since also $[A, G/G(\Gamma)] = 0$ for all torsion groups $A \in \mathfrak{R}$, $G/G(\Gamma)$ is $\mathfrak{R}$ -semisimple, so that $G(\Gamma) = \mathfrak{R}(G)$.

(iii) $\Longrightarrow$ (iv): If $G(\Gamma) = \mathfrak{R}(G)$ for every torsion-free group G , then $\mathcal{C}(\Gamma)$ is the class of torsion-free groups in $\mathfrak{R}$. Since $\mathcal{C}(\Gamma)$ is pure-hereditary, so is $\mathfrak{R}$, by [11], Theorem 3.2.

(iv) $\Longrightarrow$ (i): If $0 \neq x \in G \in \mathfrak{R}$ which is pure-hereditary and G is torsion free, then $\mathfrak{R}$ contains $[x]_*$, the smallest pure subgroup of G which contains x . Thus $T(x) = T([x]_*) \in \Gamma$ for all $x \in G$, i.e. $G \in \mathcal{C}(\Gamma)$. Thus $\mathcal{C}(\Gamma)$ is the class of torsion-free members of $\mathfrak{R}$ and is therefore closed under extensions.

(iv) $\Longleftrightarrow$ (v): [11], Theorem 4.2. //

Theorem 2.12 contains an extension of the theorem in [12].

It is a consequence of Proposition 2.2 that $L(X) = L(Y)$ for rational groups X and Y if and only if $X \cong Y$. The corresponding statement is false for groups of equal higher rank. By Theorem 1.3, if G and H are quasi-isomorphis torsion-free groups, then $L(G) = L(H)$ but the converse implication does not hold, as the following example shows.

Example 2.13. Let $(h_1, h_2, \ldots)$ be a height-sequence for which $0 < h_n < \infty$ for infinitely many values of n, τ the corresponding type, X a rational group of type τ. As shown in [12], there exists a torsion-free group G of rank 2 such that $G(\tau) \cong X \cong G/G(\tau)$. Clearly

$$L(G) = L(X) = L(X \oplus Y)$$

for any rational group Y with $T(Y) \geq \tau$ and for such Y the rank-2 groups G and $X \oplus Y$ are not quasi-isomorphic.

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(Oblatum 26.4.1972)