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A NOTE ON FRÉCHET SPACES ¹⁾

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Recall that a Fréchet space (L, \mathcal{A}) is a T_1 topological space such that for every subset A we have $\lambda A = \{x \mid x = \lim x_n, x_n \in A\}$, i.e. λA is the set of all limit points of sequences of points of A ; the space (L, \mathcal{A}) is said to be sequentially regular if for every sequence $\langle x_n \rangle$ of points of L and every point x such that $x \in L - \lambda U(x_n)$ there is a continuous function f on (L, \mathcal{A}) , $0 \leq f(x) \leq 1$, and a subsequence $\langle n_i \rangle$ of $\langle n \rangle$ such that $f(x) = 0$, $f[U(x_{n_i})] = 1$ (cf. [3]).

Following [5] a T_1 topological space (L, \mathcal{A}) is called κ_0 -regular if for every countable subset A and every point x such that $x \in L - \lambda A$ there is a continuous function f on (L, \mathcal{A}) , $0 \leq f(x) \leq 1$ such that $f(x) = 0$, $f[A] = 1$. It can be readily seen that every κ_0 -regular Fréchet space is sequentially regular. J. Novák asked in [5] whether every sequentially regular Fréchet space is κ_0 -regular.

 1) The article is a part of [1].

The main purpose of the present paper is to show that the answer is no. The space $\bar{\Lambda}_\infty$ constructed by F.B. Jones in [2] ²⁾ (as a Moore space which is not completely regular) is a counter-example. We also give a necessary and sufficient condition for a Fréchet sequentially regular space to be \mathcal{K}_0 -regular and two sufficient conditions for an \mathcal{K}_0 -regular Fréchet space to be completely regular.

Example. Let L be the subset of all points (x, y) of the Euclidean plane $\mathbb{R} \times \mathbb{R}$ such that $y \geq 0$ provided with the following refinement of the product topology: for $\kappa > 0$, the sets

$$V^\kappa(x, 0) = \{(x, 0)\} \cup \{(u, v) \mid (u, v) \in L, (u-x)^2 + (v-\kappa)^2 < \kappa^2\}$$

are also neighbourhoods of the point $(x, 0)$ (Niemytzky space).

Denote by \mathcal{A} the just described topology. Clearly, (L, \mathcal{A}) satisfies the first axiom of countability and hence it is Fréchet. The subspace $(D, \mathcal{A}/D)$ of (L, \mathcal{A}) where $D = \{(x, 0) \mid x \in \mathbb{R}\}$, is discrete. The space (L, \mathcal{A}) is completely regular and hence sequentially regular. The set D is the union of two disjoint uncountable sets, denote them by A and by B , such that if U is an open set containing uncountably many points of one of them, then $\mathcal{A}U$ contains uncountably many points of the other (for the proof see [2]).

Let $\langle (L_n, \mathcal{A}_n) \rangle_{n=1}^\infty$ be a simple sequence of disjoint copies of the space (L, \mathcal{A}) . For convenience we may

 2) It is Professor J. Novák who called my attention to that article.

imagine these spaces as lying in different planes of the three-dimensional Euclidean space parallel to the plane of L . For each point set H in L and to every natural m there corresponds in a natural way the set H_m in L_m (the set H is the projection of every H_m). The symbol q denotes always a point of D .

Let $\sum_{n=1}^{\infty} (L_n, \lambda_n)$ be the topological sum of the above sequence. We modify it in the following manner:

1. If n is odd ($n = 1, 3, 5, \dots$) and q is a point of B , then we identify points q_n and q_{n+1} to $(q_n; q_{n+1})$; if n is even ($n = 2, 4, 6, \dots$) and q is a point of A , then we identify points q_n and q_{n+1} to $(q_n; q_{n+1})$ (the projection of $(q_n; q_{n+1})$ is q in this case). Let for $\kappa > 0$ the sets

$$W^{\kappa}((q_n; q_{n+1})) = \\ = \{(q_n; q_{n+1})\} \cup \{V_n^{\kappa}(q) - (q)\} \cup \{V_{n+1}^{\kappa}(q) - (q_{n+1})\}$$

be fundamental systems of neighbourhoods of these points, i.e. we take a quotient space of $\sum_{n=1}^{\infty} (L_n, \lambda_n)$.

2. We add one "ideal" point μ (distinct from all) to the modified $\sum_{n=1}^{\infty} (L_n, \lambda_n)$.

Let for $k = 1, 2, 3, \dots$, the sets

$$O_k(\mu) = \{\mu\} \cup \left\{ \bigcup_{m \geq k} \bigcup_{\eta > 0} (x_m, y_m) \right\} \cup \left\{ \bigcup_{m \geq k} (q_m; q_{m+1}) \right\}$$

form a fundamental system of neighbourhoods of μ .

Denote by $(L_{\infty}, \lambda_{\infty})$ this modified space (cf. [2], where $\lambda_{\infty} = (L_{\infty}, \lambda_{\infty})$). The space $(L_{\infty}, \lambda_{\infty})$ satisfies the first axiom of countability and hence it is

Fréchet, it is "completely regular at every point" except ρ but it is not completely regular (at ρ) since $\rho \in L_\infty - \lambda_\infty A_1$, but for each continuous function f on $(L_\infty, \lambda_\infty)$ we have $f(\rho) \in \overline{f[A_1]}$ (cf. [2]).

Proposition. The Fréchet space $(L_\infty, \lambda_\infty)$ is sequentially regular but fails to be κ_0 -regular.

Proof. First prove that $(L_\infty, \lambda_\infty)$ is sequentially regular. Since $(L_\infty, \lambda_\infty)$ is "completely regular and hence sequentially regular at every point" except ρ , we have to prove that if $\langle x_m \rangle$ is a sequence of points of L_∞ such that $\rho \in L_\infty - \lambda_\infty \bigcup_{m=1}^{\infty} (x_m)$, then there is a continuous function f on $(L_\infty, \lambda_\infty)$ and a subsequence $\langle x_{m_i} \rangle$ of $\langle x_m \rangle$ such that

$$f(\rho) = 0, \quad f(x_{m_i}) = 1, \quad i = 1, 2, 3, \dots$$

Since there is a natural κ_0 such that $x_m \in L_\infty - \mathcal{O}_{\kappa_0}(\rho)$ for all m , we always can and do select a subsequence $\langle x'_{m_i} \rangle$ of $\langle x_m \rangle$ such that

a) $\langle x'_{m_i} \rangle$ is a constant sequence or the projection of no x'_{m_i} lies in $\mathcal{D} \subset L$. In this case the construction of f and the subsequence $\langle x_{m_i} \rangle$ of $\langle x'_{m_i} \rangle$ and hence of $\langle x_m \rangle$ is easy and is omitted.

b) If $(x'_i, 0) \in \mathcal{D} \subset L$ is the projection of x'_{m_i} , i.e. x'_{m_i} is either of the form of $(q_m^{(i)}; q_{m+1}^{(i)})$, $m \leq \kappa_0$, or $x'_{m_i} \in A_1$, then there is a strictly monotone, say increasing, subsequence $\langle x_i \rangle$ of the sequence $\langle x'_i \rangle$ of real numbers x'_i . Let $\langle \kappa_i \rangle$ be a sequence of positive real numbers such that

$$x_{i-1} + \kappa_{i-1} < x_i - \kappa_i < x_i + \kappa_i < x_{i+1} - \kappa_{i+1}, \quad i = 1, 2, 3, \dots$$

Denote by $U(x_{m_i}) = (V^{\kappa_i}(x_i, 0))_1$ if $x_{m_i} \in A_1$ and

$$U(x_{m_i}) = W^{\kappa_i}((q_n^{(i)}; q_{n+1}^{(i)}))$$

otherwise. Now, let f be a function on $(L_\infty, \lambda_\infty)$ defined in the following manner:

$$f(x) = 1 \text{ for } x = x_{m_i};$$

$f(x) = 0$ for each x on the boundary of the neighbourhood $U(x_{m_i})$ of x_{m_i} and linear on the segment from x_{m_i} to x , $i = 1, 2, 3, \dots$;

$$f(x) = 0 \text{ for } x \in L_\infty - \bigcup_{i=1}^{\infty} U(x_{m_i}).$$

It is easy to verify that f has the desired properties. If the sequence $\langle x_i \rangle$ is decreasing, then the procedure is similar.

Secondly, denote by

$$C = \{f(x, y) \mid (x, y) \in L - D; x, y \text{ rational}\}.$$

The set C_1 is countable and can be arranged into a sequence $\langle x_m \rangle$ and $\rho \in L_\infty - \bigcup_{m=1}^{\infty} U(x_m)$. As $A_1 \subset \bigcup_{m=1}^{\infty} U(x_m)$, we have $f(\rho) \in \overline{\bigcup_{m=1}^{\infty} (f(x_m))}$ for each continuous function f on $(L_\infty, \lambda_\infty)$. Therefore $(L_\infty, \lambda_\infty)$ fails to be κ_0 -regular. This completes the proof.

Let (L, λ) be a Fréchet sequentially regular space. Recall that the completely regular modification $\tilde{\lambda}$ of λ is the finest of all completely regular topologies for L coarser than λ , the systems of continuous functions on

(L, λ) and on $(L, \tilde{\lambda})$ coincide and $\lim x_m = x$ if and only if the sequence $\langle x_m \rangle$ is eventually in every $\tilde{\lambda}$ -neighbourhood of x (see [3]). A point x_0 is called a side-point of a sequence $\langle x_m \rangle$ in $(L, \tilde{\lambda})$ if any subsequence $\langle x_{m_i} \rangle$ of $\langle x_m \rangle$ does not converge to x_0 and the sequence $\langle x_m \rangle$ is frequently in every $\tilde{\lambda}$ -neighbourhood of x_0 .

Theorem 1. A Fréchet sequentially regular space (L, λ) is x_0 -regular if and only if there is no sequence in $(L, \tilde{\lambda})$ having a side-point, where $\tilde{\lambda}$ is the completely regular modification of λ .

Proof. I. If there is a sequence $\langle x_m \rangle$ in $(L, \tilde{\lambda})$ having a side-point x_0 , then

$$x_0 \in L - \lambda U(x_m), \quad x_0 \in \tilde{\lambda} U(x_m).$$

Thus for each continuous function f on $(L, \tilde{\lambda})$ and hence, as mentioned above, on (L, λ) we have

$$f(x_0) \in \overline{U(f(x_m))}.$$

But this implies that (L, λ) cannot be x_0 -regular.

II. If (L, λ) is not x_0 -regular, then there is a sequence $\langle x_m \rangle$ of points $x_m \in L$ and a point $x_0 \in L$ such that

$$x_0 \in L - \lambda U(x_m)$$

and for each continuous function f on (L, λ) there is a subsequence $\langle m_i \rangle$ of $\langle m \rangle$ such that

$$\lim f(x_{m_i}) = f(x_0).$$

From the definition of $\tilde{\lambda}$ it follows that

$$x_0 \in \tilde{\mathcal{U}}(x_m) ,$$

i.e. x_0 is a side-point of the sequence $\langle x_m \rangle$ in $(L, \tilde{\mathcal{A}})$.

Theorem 2. A regular separable \mathcal{X}_0 -regular Fréchet space (L, \mathcal{A}) is completely regular.

Proof. Denote by $S \subset L$ a countable set such that $G \cap S \neq \emptyset$ for each non-empty open set $G \subset L$. Let $F \subset L$ be a non-empty closed set and $x_0 \in L - F$. Then there is a neighbourhood $W(x_0)$ such that $\mathcal{A}W(x_0) \subset L - F$ and $(L - \mathcal{A}W(x_0)) \cap S \neq \emptyset$. Hence $(L - W(x_0)) \cap S \neq \emptyset$. Now, arrange the countable set $(L - W(x_0)) \cap S$, either finite or infinite, into a sequence $\langle x_m \rangle$. Evidently

$$x_0 \in (L - \mathcal{A}U(x_m)) \subset L - F .$$

Since (L, \mathcal{A}) is \mathcal{X}_0 -regular, there is a continuous function f on (L, \mathcal{A}) such that

$$f(x_0) = 0, f[U(x_m)] = 1 = f[F] .$$

Corollary. A first-countable separable \mathcal{X}_0 -regular topological space is completely regular.

Proof. Professor J. Novák proved in [4] that every first-countable sequentially regular topological space is regular. The assertion follows at once from the foregoing Theorem 2.

R e f e r e n c e s

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