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ON THE CANONICAL SUBDIRECT DECOMPOSITION OF A JOIN SEMI-LATTICE

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1. Introduction. By a subdirect union of the algebras A_p ($p \in P$) a subalgebra R of the direct union $\Pi(A_p; p \in P)$ is meant, having the property that $f_p(R) = A_p$ for every decomposition homomorphism f_p of $\Pi(A_p; p \in P)$. It is said that the algebra A can be represented as the subdirect union of the algebras A_p if A is isomorphic to a subdirect union of the A_p; this subdirect union is called the subdirect decomposition of A with factors A_p. An algebra is called subdirectly decomposable or subdirectly reducible if A has a subdirect decomposition, no decomposition homomorphism of which is an isomorphism. Further let A be an algebra and P a set of indices. The algebra A can be represented as a subdirect union of some algebras A_p, $p \in P$, if and only if A has congruence relations $(\theta_p; p \in P)$ such that $\bigcap(\theta_p; p \in P) = 0$, the equality relation (see e.g. [1, Cor. 1, p. 140]).

Let the algebra A be a lattice L or a join semi-

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lattice $L_\cup$, and $\Theta(A)$ the lattice of all congruence relations on A . For any element $\theta \in \Theta(A)$ there exists in $\Theta(A)$ and element θ^* called the pseudocomplement of θ . The correspondence $\theta \longrightarrow \theta^{**}$ is a closure operation on $\Theta(A)$ and the closed elements $\theta^{**} = \theta$ form a complete boolean algebra $\Theta_*(A)$ on which the join operation is given by $\theta \vee \Phi = (\theta \cup \Phi)^{**}$ (when $A = L_\cup$, see [4, Thm.4]).

Let $\{\theta_p ; p \in P\}$ be a subset of $\Theta_*(A)$ such that $\theta_p^* = \bigcap(\theta_q ; q \in P, q \neq p)$ for all $p \in P$, then $\bigcap(\theta_p ; p \in P) = \theta_p \cap \theta_p^* = 0$ and thus the set $\{\theta_p ; p \in P\}$ generates a subdirect decomposition of A . Such a decomposition is called canonical by F. Maeda [3]. In order that the set $\{\theta_p ; p \in P\}$ generates a canonical subdirect decomposition of an algebra A , it is necessary and sufficient that $\theta_p \in \Theta_*(A)$ for every $p \in P$, $\bigcap(\theta_p ; p \in P) = 0$, and $\theta_q \vee \theta_p = 1$ $(p \neq q)$. The proof for $A = L_\cup$ is obvious according to the proof of F. Maeda in the case $A = L$ (see [3, Thm. 2.1]).

As pointed out by T. Tanaka [5, Remark 1], if $\theta_p^* = \bigcap(\theta_q ; q \in P, q \neq p) = 0$, then $\theta_p = \theta_p^{**} = 1$ and the factor corresponding to θ_p can be omitted.

2. <u>On the canonical subdirect decomposition of a semilattice with finite number of factors.</u> In the following we shall consider the structure of a semilattice $L_\cup$ having

a canonical subdirect decomposition with finite number of simple factors $L_{p\cup}$, i.e., every $\theta(L_{p\cup})$ contains exactly two elements. Thus every factor $L_{p\cup}$ corresponds to a maximal congruence relation θ^o_p on L.

According to D. Papert [4, Thm. 1], every maximal congruence relation θ^o on $L_\cup$ is given by an ideal I of $L_\cup$ such that $x\,\theta^o_I\,y$ if and only if $x, y \in I$, or $x, y \notin I$.

The notation $a \prec b$, $a, b \in L_\cup$, means that if there is an element $c \in L_\cup$ such that $c > a$ and c is comparable with b, then $c \geq b$. One calls b an immediate successor of a. We denote by $is(a)$ the set of immediate successors of a. $|is(a)|$ implies the number of the elements in the set $is(a)$.

<u>Lemma 1</u>. If a semilattice $L_\cup$ is finite and C a set of elements of $L_\cup$ having the property $c \in C$, $|is(c)| = 1$, then every maximal congruence relation $\theta^o_{(a]}$, $a \in C$, on $L_\cup$ has a complement $(\theta^o_{(a]})'$ in $\theta(L_\cup)$, where $(a]$ is a principal ideal of $L_\cup$ generated by a.

<u>Proof</u>. Let 1_θ and 0_θ be the greatest and the least element of the lattice $\theta(L_\cup)$, respectively. We shall show that $(\theta^o_{(a]})' = \bigcap(\theta^o_{(c]}\,;\ c \in C,\ c \neq a)$, where $a \in C$.

At first we show that $\bigcap(\theta^o_{(c]}\,;\ c \in C) = 0_\varepsilon$ The relation before is valid if (1) for every $b \in L_\cup$, $b \neq 1 \in L_\cup$, $b \in (c]$ for some $c \in C$, and (2) if for every two disjoint elements $b_1, b_2 \in L_\cup$, $b_1, b_2 \neq 1$, there is an element $c \in C$ such that $b_1 \in (c]$ and $b_2 \notin (c]$. The condition (1) follows immediately from the fact that for every element $k \in L_\cup$, $k \prec 1$, $|is(k)| = 1$.

(2) b_1 and b_2 can be (i) comparable, or (ii) non-comparable. (i) If b_1 and b_2 are comparable, then we can assume without any loss of generality, $b_1 < b_2$. According to the finity of $L_\cup$, there is in $L_\cup$ a finite chain $b_1 = x_0 \prec x_1 \prec x_2 \prec \dots \prec x_m = b_2$. If for some x_j, $j = 0, \dots, m-1$, $|is(x_j)| = 1$, the assertion is immediately valid. If $|is(x_j)| \geq 2$, we can choose an immediate successor $y_1 \neq x_1$ for $b_1 = x_0$, and if $|is(y_1)| = 1$, the assertion follows. If $|is(y_1)|' \geq 2$, then, after a finite number of similar steps, we can reach an element $c \in C$ for which the assertion is valid, since $L_\cup$ is finite. In the case (ii), where b_1 and b_2 are not comparable, $b_1 \cup b_2 > b_1, b_2$. Then according to (i) above we find an element $c \in C$ such that say $b_1 \in (c]$ and $b_1 \cup b_2 \notin (c]$. But then $b_2 \notin (c]$, since if $b_2 \in (c]$, so $b_1 \cup b_2 \in (c]$, which is a contradiction.

Trivially, $1 \notin C$. Then obviously $a \,(\bigcap(\theta^0_{(c]}\,;\ c \in C, c \neq a))\, d$, where $d = is(a)$ and thus $\theta^0_{(a]} \cup \bigcap(\theta^0_{(c]}\,;\ c \in C, c \neq a) = 1_\theta$. Hence

$$(\theta^0_{(a]})' = \bigcap(\theta^0_{(c]}\,;\ c \in C, c \neq a).$$

<u>Theorem 1.</u> Every finite semilattice L has a canonical subdirect decomposition with simple factors.

The proof follows directly from Lemma 1 and its proof.

Theorem 1 shows that a canonical subdirect decomposition of a semilattice $L_\cup$ with finite number of simple factors does not imply any structural properties for $L_\cup$ different from the case of lattices (see Dilworth [2, Thm. 3.3]).

3. An infinite construction. In the following, we consider a class of infinite semilattices which has a canonical subdirect decomposition with simple factors. We shall call a semilattice $L_\cup$, for which $\Theta(L_\cup)$ is distributive, a quasidistributive semilattice. D. Papert has proved [4, Thm. 7] that a semilattice $L_\cup$ is quasidistributive if and only if any two noncomparable elements of $L_\cup$ have no lower bound in $L_\cup$.

Lemma 2. Let $L_\cup$ be a semilattice, $a, b \in L_\cup$, $a \neq b$, and θ_{ab} a binary relation on $L_\cup$ such that $x \theta_{ab} y$ if and only if (i), or (ii) and (iii) are valid, where (i) $x = y$, (ii) $a \cup b \cup x = a \cup b \cup x \cup y = a \cup b \cup y$; (iii) $a \cup x = x$ or $b \cup x = x$ and $a \cup y = y$ or $b \cup y = y$. Then θ_{ab} is a minimal congruence relation on $L_\cup$ collapsing the elements a and b of $L_\cup$.

The proof is obvious.

Following J. Varlet [6] we define a part of a semilattice $L_\cup$. Let $a, b \in L_\cup$, $a \neq b$. The part $\langle a, b \rangle$ of $L_\cup$ is a set-theoretical union of the elements of $L_\cup$ contained by the closed intervals $[a, a \cup b]$ and $[b, a \cup b]$ of $L_\cup$.

We shall say that a congruence class C modulo θ is trivial if for any two elements $x, y \in C$, $x \sim y$.

<u>Lemma 3</u>. A semilattice $L_\cup$ is quasidistributive if and only if the only nontrivial congruence class of the congruence relation θ_{ab} is the part $\langle a, b \rangle$ of $L_\cup$.

<u>Proof</u>. 1° Let $L_\cup$ be a quasidistributive semilattice and $c\,\theta_{ab}\,d$, $c, d \notin \langle a, b \rangle$, $a \neq b$ and $c \neq d$, and $a, b, c, d \in L_\cup$. According to the definition of θ_{ab} only three cases arise: (i) $c \cup d > a \cup b$, (ii) $c \cup d < a \cup b$, and (iii) $c \cup d$ and $a \cup b$ are noncomparable.

(i) $c\,\theta_{ab}\,d \Longleftrightarrow c\,\theta_{ab}\,c \cup d$ and $d\,\theta_{ab}\,c \cup d$. Thus $a \cup c \cup d = c \cup d = b \cup c \cup d$. But if c (or d) is noncomparable with $a \cup b$, then $a \cup c \neq c$ and $b \cup c \neq c$ ($a \cup d \neq d$ and $b \cup d \neq d$), since $a \cup b$ and c (d) have not a common lower bound in $L_\cup$ (see [4, Thm. 7]). If for c (or d), $c > a \cup b$, then $c \cup a \cup b \neq a \cup b \cup c \cup d$ (or $d \cup a \cup b \neq a \cup b \cup c \cup d$), since $d \neq c$. Hence $c\,\not\theta_{ab}\,d$.

(ii) If $c \cup d < a \cup b$, then $a \cup c \neq c$ and $c \cup b \neq c$, since if $c \cup a = c$ or $c \cup b = c$, then $c \in \langle a, b \rangle$, which is a contradiction.

(iii) $a \cup c = c$, $b \cup c \neq c$, since the noncomparable elements have not a common lower bound in $L_\cup$.

2° Let the only nontrivial congruence class module θ_{ab} be the part $\langle a, b \rangle$ of $L_\cup$ for every two elements a, $b \in L_\cup$. Assume that two noncomparable elements c and d of $L_\cup$ have a common lower bound h in $L_\cup$ (see [4, Thm.

7]), and consider the congruence relation $\theta_{hc} \cdot d\theta_{hc}\, c \cup d$, since $h \cup d = d$, $c \cup d \cup c = c \cup d$, and $d \cup h \cup c = d \cup c \cup h \cup c$. But $d \notin \langle h, c \rangle = [h, c]$, since d and c are noncomparable, and $d \cup c \notin [h, c]$, since $c < d \cup c$. Thus $d\,\theta_{hc}\, c \cup d$ implies a contradiction.

Now we can prove a theorem concerning the complement of θ_{ab} in $\theta(L_\cup)$.

Lemma 4. If $L_\cup$ is a quasidistributive semilattice, then for any two elements $a, b \in L_\cup$, $a \neq b$, θ_{ab} has a complement θ'_{ab} in $\theta(L_\cup)$.

Proof. Consider the congruence relation $\bigcap_{x \in A} \theta^{o}_{(x]} = X$, where $A = \langle a, b \rangle - a \cup b$. The congruence relation exists, since $\theta(L_\cup)$ is the complete lattice. If $z(\theta_{ab} \cap X)u$, where $z \neq u$, $z, u \in L_\cup$, then $z\,\theta_{ab}\,u$ and according to Lemma 3, $z, u \in \langle a, b \rangle$. This implies $\theta^{o}_{(z]} \in \{\theta^{o}_{(x]} : x \in A\}$ for which $z\, \not\theta^{o}_{(z]}\, z \cup u$, which is a contradiction. Hence $\theta_{ab} \cap X = 0_\theta$.

Consider $\theta_{ab} \cup X$. Let $z \neq u$ be two elements of $L_\cup$. We show that $u(\theta_{ab} \cup X)z \cup u$ which implies $\theta_{ab} \cup X = 1_\theta$. The proof contains three cases: (i) $u \geq a \cup b$, (ii) u and $a \cup b$ are noncomparable, and (iii) $u < a \cup b$.

(i) If $u \geq a \cup b$, then $u \cup z \geq a \cup b$ and $u\,\theta^{o}_{(x]}\, z \cup u$ for every $x \in A$.

(ii) If u and $a \cup b$ are noncomparable, then $z \cup u \nleq a \cup b$, since $u \nleq a \cup b$, and thus $z \cup u \notin \langle a, b \rangle$.

Then $u\,\theta^{o}_{(x]}\,z \cup u$ for every $x \in A$.

(iii) If $u < a \cup b$, then (1) $u \in \langle a, b\rangle$ or (2) $u < a$ (or $u < b$), or (3) $u < a \cup b$ and u is noncomparable with a and b . (1) If u , $z \cup u \in \langle a, b\rangle$, then $u\,\theta_{ab}\,z \cup u$ and if $z \cup u \notin \langle a, b\rangle$ then $z \cup u > a \cup b$, since two noncomparable elements have not a common lower bound in $L_{\cup}$, and thus $u\,\theta_{ab}\,a \cup b$ and $a \cup b\,\theta^{o}_{(x]}\,z \cup u$ for every $x \in A$. (2) If $u < a$,then $u\,\theta^{o}_{(x]}\,a$ for every $x \in A$, for $u \in (x]$ if and only if $a \in (x]$, since two noncomparable elements of $L_{\cup}$ have not a common lower bound in $L_{\cup}$. The last part of the proof is similar to that of (1). (3) $u < a \cup b$ and u is noncomparable with a and b , then $u \notin \langle a, b\rangle$. Thus $u\,\theta^{o}_{(x]}\,u \cup b$ or $u\,\theta^{o}_{(x]}\,u \cup a$ for every $x \in A$ and further $u \cup b\,\theta_{ab}\,a \cup b$ (or $u \cup a\,\theta_{ab}\,a \cup b$). After this we can continue as in the case (1). Hence X is the complement of θ_{ab} in $\theta(L_{\cup})$.

<u>Theorem 2</u>. Let $L_{\cup}$ be a quasidistributive semilattice, where for every element $a \in L_{\cup}$, $a \neq 1$, there exists an element $b \in is(a)$. Then $L_{\cup}$ has a canonical subdirect decomposition with simple factors if and only if $1 \in L_{\cup}$.

<u>Proof</u>. 1° Let $1 \in L_{\cup}$. Clearly $\bigcap(\theta^{o}_{(x]}\,;\,x \in C) = 0_{\theta}$, where $C = L_{\cup} - 1$. It follows from the quasidistributivity of $L_{\cup}$ that for every $a \neq 1$, $|is(a)| = 1$. Thus the assumption of the theorem well defines the set $is(a)$. But then $a\,(\bigcap(\theta^{o}_{(x]}\,;\,x \in C,\ x \neq a))\,b = is(a)$ which

implies $\theta^{o}_{(a]} \cup \bigcap(\theta^{o}_{(x]}\ ;\ x \in C,\ x \neq a) = 1_{\theta}$, and the theorem follows.

2^{o}. Let the set $\{\theta^{o}_{I_p}\ ;\ p \in P\}$ generate a canonical subdirect decomposition of $L_{\cup}$ with simple factors. According to Remark 1 of T. Tanaka [5] $L_{\cup} \notin \{I_p\ ;\ p \in P\}$, and thus the set $D = \{d : d \notin I_p$ for any $p \in P, d \in L_{\cup}\}$ is nonempty. If $|D| \geq 2$, then $\bigcap(\theta^{o}_{I_p}\ ;\ p \in P) \neq 0_{\theta}$, which is a contradiction. Hence $D = \{d\}$. If $L_{\cup}$ contains an element a, $a > d$ or a is noncomparable with d, then $d \in I_p$ for some $p \in P$, since $a \in I_p$, and $a \cup d \in I_{p'}$, $p, p' \in P$; a contradiction. Thus $d \geq a$ for every $a \in L_{\cup}$, whence $1 \in L_{\cup}$.

Lemmas 2, 3 and 4 form a part of the work [7].

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