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HAUSDORFF MEASURES OF THE SET OF CRITICAL VALUES OF FUNCTIONS OF THE CLASS $C^{k,\lambda}$

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This paper deals with the problem of critical values of real functions. The following assertion is known for functions of one variable (see [1]): If f is a function of the class $C^{k,\lambda}$, then $\mu_s(f(Z)) = 0$, where $s = \frac{1}{k+\lambda}$, μ_s is a s - Hausdorff measure and Z denotes the set of all critical points of the function f. In this paper there is proved an analogous assertion for functions defined on some open set in E_n. Theorem 4.2 and Remark 4.1 give a full answer to the question how big the set of critical values can be in dependence of the smoothness of our function f. This result is proved for $\lambda = 0$ (i.e. for $f \in C^k$) in [2],[3],[4].

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1. <u>Notations and terminology</u>. We shall denote by Ω a fixed open set in the n-dimensional Euclidean space E_n. Let k be a positive integer number, $\lambda \in \langle 0,1 \rangle$, let f be a function defined on Ω. Then we write $f \in C^{k,\lambda}(\Omega)$ if

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f has on Ω continuous derivatives of all orders not exceeding k and if derivatives of the order k are λ-Hölderian. We shall denote the set of critical points of the given function by $Z = \{x \in \Omega ; \frac{\partial f}{\partial x_i}(x) = 0, i = 1, \dots, m\}$. If $\beta = (\beta_1, \beta_2, \dots, \beta_m)$ is a multiindex then we write $|\beta| = \beta_1 + \dots + \beta_m$ and $D^\beta f = \frac{\partial^{|\beta|} f}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \dots \partial x_m^{\beta_m}}$. Suppose ψ is a mapping defined on a domain D in E_d, the range of which lies in E_m. We denote by $\psi_1, \dots, \psi_m$ the components of this mapping and write $\psi \in C^{k,\lambda}(D)$ if $\psi_i \in C^{k,\lambda}(D)$.

The composition of the function f and of the mapping ψ is denoted by $f * \psi$, the derivative of this composition is denoted by $D^\beta(f * \psi)$; the symbol $D^\beta f * \psi$ denotes the composition of the function $D^\beta f$ and of ψ.

If $x = (x_1, \dots, x_m) \in E_m$, then we put $\|x\| = \left(\sum_{i=1}^m x_i^2\right)^{\frac{1}{2}}$. By $D(x)$ we denote an open ball with the center in the point x. If $x^0 \in E_m$, then by $\overline{xx^0}$ we denote an open segment with the extreme points x, x^0.

2. General remarks

Remark 2.1. Let $F_1, \dots, F_s \in C^{k,\lambda}(\Omega)$ be functions, $x^0 \in \Omega$. Suppose, for each $i = 1, \dots, s$, there exists j such that $\frac{\partial F_i}{\partial x_j}(x^0) \neq 0$, $F_i(x^0) = 0$. Denote $N = \{x \in \Omega ; F_i(x) = 0$ for each $i = 1, \dots, s\}$. Then there exists a number $d < m$, the balls $D(x^0) \subset \Omega$, $D(y^0) \subset E_d$ and a mapping $\Phi \in C^{k,\lambda}(D(y^0))$ such that $\Phi(y^0) = x^0$, $N \cap D(x^0) \subset \Phi(D(y^0)) \subset \Omega$ and such that either $d = 1$ or

(1) $$\frac{\partial}{\partial y_j}(F_i * \Phi)(y^0) = 0 \quad \text{for each } i = 1,\dots,s\,; \quad j = 1,\dots,d\,.$$

Proof. We can choose a submatrix I of the matrix $M = \left(\frac{\partial F_i}{\partial x_j}(x^0)\right)_{i=1,\dots,s}^{j=1,\dots,m}$ with the following properties: $\det I \neq 0$ and rank $I = \max(\operatorname{rank} S)$, where maximum is taken over all submatrices S of M such that $\operatorname{rank} S < m$. We can suppose

$$I = \left(\frac{\partial F_i}{\partial x_j}(x^0)\right)_{i=1,\dots,r}^{j=1,\dots,r}, \text{ where } 0 < r < m\,,\ r \leq s\,.$$

From the implicit function theorem it follows that there exist the balls $D(x^0) \subset \Omega$, $D(y^0) \subset E_d$, where $d = m - r$ and the functions $\varphi_1,\dots,\varphi_r \in C^{k,\lambda}(D(y^0))$ such that

(2) $F_i(\varphi_1(y),\dots,\varphi_r(y), y_1,\dots,y_{m-r}) = 0$

for $i = 1,\dots,r$, $y = (y_1,\dots,y_{m-r}) \in D(y^0)$,

(3) if $x \in D(x^0)$, $x \in N$, then $x_i = \varphi_i(x_{r+1},\dots,x_m)$

for $i = 1,\dots,r$.

Define $\Phi(y) = (\varphi_1(y),\dots,\varphi_r(y), y_1,\dots,y_d)$ for

$y = (y_1, \dots, y_d) \in D(y^0)$. By (3) we have $N \cap D(x^0) \subset \Phi(D(y^0))$. The condition (1) for $i = 1, \dots, r$ follows from (2). If $d > 1$, then $rank\, M = r$ and the vectors $\left(\frac{\partial F_i}{\partial x_1}(x^0), \dots, \frac{\partial F_i}{\partial x_m}(x^0)\right)$ for $i = r+1, \dots, s$ are linear combinations of

$\left(\frac{\partial F_i}{\partial x_1}(x^0), \dots, \frac{\partial F_i}{\partial x_m}(x^0)\right)$ for $i = 1, \dots, r$.

From here the condition (1) follows for $i = r+1, \dots, s$ too.

Remark 2.2. Let $F \in C^{\ell}(\Omega)$ be a function, $x^0 \in \Omega$, $D^{\beta} F(x^0) = 0$ for each $0 < |\beta| \leqq \ell - 1$. Suppose D is a ball in E_d, $d \leqq n$. Let $\psi \in C^{(1)}(D)$ be a mapping, $\psi(D) \subset \Omega$, $z^0 \in D$, $\psi(z^0) = x^0$. Denote

$$C_1 = \max_{\substack{i = 1, \dots, n \\ j = 1, \dots, d}} \left(\sup_{z \in D} \left| \frac{\partial \psi_i}{\partial z_j}(z) \right| \right) < +\infty .$$

Then for each $z \in D$ there exists $z^1 \in \overline{z z^0}$ and $C > 0$ (C depends on C_1 and ℓ only) such that

$$|F(\psi(z)) - F(\psi(z^0))| \leqq C \cdot \sum_{|\beta| = \ell} |D^{\beta} F(\psi(z^1))| \cdot \|z - z^0\|^{\ell} .$$

Proof. There exists $z^1 \in \overline{z z^0}$ such that

$$|F(\psi(z)) - F(\psi(z^0))| = \left|\sum_{j=1}^{d} \frac{\partial}{\partial z_j}(F * \psi)(z^1) \cdot (z_j - z_j^0)\right| =$$

$$= \left|\sum_{j=1}^{d} \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(\psi(z^1)) \cdot \frac{\partial \psi_i}{\partial z_j}(z^1) \cdot (z_j - z_j^0)\right| \leqq$$

$$\leq C_1 \cdot \sum_{i=1}^{n} \left| \frac{\partial F}{\partial x_i} (\psi(x^1)) \right| \cdot \| x - x^0 \| .$$

In a similar way we can estimate

$$\left| \frac{\partial F}{\partial x_i} (\psi(x^1)) \right| = \left| \frac{\partial F}{\partial x_i} (\psi(x^1)) - \frac{\partial F}{\partial x_i} (\psi(x^0)) \right| \leq$$

$$\leq C_1 \sum_{t=1}^{n} \left| \frac{\partial^2 F}{\partial x_i \, \partial x_t} (\psi(x^2)) \right| \cdot \| x^1 - x^0 \|$$

where $\| x^0 - x^1 \| \leq \| x - x^0 \|$. Further we can estimate $\frac{\partial^2 F}{\partial x_i \, \partial x_t} (\psi(x^2))$ etc. After a finite number of steps we obtain our assertion.

Remark 2.3. (Hausdorff measure.) Suppose A is a subset in E_m and s is a positive real number. For each $\varepsilon > 0$ define $\mu_{s,\varepsilon}(A) = \inf \sum_{i=1}^{\infty} (\operatorname{diam} A_i)^s$, the infimum being taken over all countable coverings $\{A_i\}_{i=1}^{\infty}$ of A such that $\operatorname{diam} A_i < \varepsilon$. The number $\mu_s(A) = \lim_{\varepsilon \to 0+} \mu_{s,\varepsilon}(A)$ is said to be s-Hausdorff measure of A . If $\mu_s(A) = 0$, then we say A is s-null.

It is easy to see: if A is s - null, then A is r-null for each $r > s$. If $s = m$, then we obtain Lebesgue measure.

3. Some estimates for functions of the class $C^{k,\lambda}(\Omega)$

Theorem 3.1. Let $f \in C^{k,\lambda}(\Omega)$ be a function. Then there exists a countable system of sets $\{M_t\}_{t=1}^{\infty}$ such that

(4) $Z \setminus \bigcup_{t=1}^{\infty} M_t$ is countable;

(5) for each positive integer t there exists $C_t > 0$ such that $|f(x_1) - f(x_2)| \leq C_t \|x_1 - x_2\|^{k+\lambda}$ for each $x_1, x_2 \in M_t$.

Remark 3.1. A similar assertion is proved in [2], but for $\lambda = 0$ only. A.P. Morse proves it by using induction for $n + k$. Theorem 3.1 can be proved in a similar way. But in this paper, a constructive proof is given. This proof is based on the fact that each set M_t lies in some hyperplane; this hyperplane is characterized by the mapping $\Phi = \Phi_1 * \dots * \Phi_p$ (on some neighborhood of a point x^0) from Construction 3.1 and Lemma 3.1; the number d_p is the dimension of this hyperplane.

Construction 3.1. Suppose $x^0 \in Z$ is a fixed point. We shall associate a finite number of mappings $\Phi_1, \dots, \Phi_p$ to this point.

Let k_1 be the smallest entire number such that $D^\beta f(x_0) = 0$ for all $|\beta| \leq k - k_1$. If $k_1 = 0$, then we need not any mapping, that means our hyperplane (see Remark 3.1) has dimension n.
Assume $k_1 > 0$. Then $\frac{\partial}{\partial x_j} D^\beta f(x^0) \neq 0$ for some $j, \beta, 1 \leq j \leq n, |\beta| = k - k_1$. Denote $Z_{k_1} = \{x \in Z; D^\beta f(x) = 0$ for all $|\beta| \leq k - k_1\}$. From the implicit function theorem it follows that there exist the balls

$D(x^0) \subset \Omega$, $D(y^0) \subset E_{d_1}$, $(d_1 < n)$ and a mapping $\Phi_1 \in C^{k_1,\lambda}(D(y^0))$ such that

(6) $Z_{k_1} \cap D(x^0) \subset \Phi_1(D(y^0)) \subset \Omega$, $\Phi_1(y^0) = x^0$

and such that either $d_1 = 1$ or

$$(7) \qquad \frac{\partial}{\partial y_j}(D^\beta f * \Phi_1)(y^0) = 0$$

for each $|\beta| = k - k_1$, $j = 1, \dots, d_1$

(see Remark 2.1; we set $F_i = D^{\alpha^i} f$, where α^i, $i = 1, \dots, s$ are all nullindexes such that $|\alpha^i| = k - k_1$,

$\frac{\partial}{\partial x_j} D^{\alpha^i} f(x^0) \neq 0$ for some j). Define $D_1 = D(y^0)$.

If $d_1 = 1$, then we set $p = 1$ and we conclude our construction.

Suppose $d_1 > 1$. Let k_2 be the smallest number such that $k_2 < k_1$ and

$$(8) \qquad D^\beta(D^{\beta^1} f * \Phi_1)(y) = 0$$

for each $|\beta^1| = k - k_1$, $|\beta| \leq k_1 - k_2$

for $y = y^0$ (β denotes d_1-dimensional multiindex in (8)).

If $k_2 = 0$, then we set $p = 1$ and we conclude our construction.

Suppose $k_2 > 0$ and denote

$$Z_{k_1,k_2} = \{x \in Z_{k_1} ; x = \Phi_1(y) , (8) \text{ is valid}\} .$$

We have $\frac{\partial}{\partial y_j} D^{\beta}(D^{\beta^1} f * \Phi_1)(y^0) \neq 0$ for some β^1, β, j, $|\beta^1| = k - k_1$, $|\beta| = k_1 - k_2$, $1 \leq j \leq d_1$.

We can, by using implicit function theorem (analogously as in the case of Φ_1 - see Remark 2.1) construct the balls $D(x^0) \subset \Omega$, $D_2 \subset E_{d_2}$, $(d_2 < d_1)$ and a mapping $\Phi_2 \in C^{k_2,\lambda}(D_2)$ such that

$$(6') \quad Z_{k_1,k_2} \cap D(x^0) \subset \Phi_1 * \Phi_2(D_2) \subset \Omega, \ \Phi_2(v^0) = y^0$$

and such that either $d_2 = 1$ or

$$(7') \qquad \frac{\partial}{\partial v_j} (D^{\beta}(D^{\beta^1} f * \Phi_1) * \Phi_2)(v^0) = 0$$

for each $|\beta^1| = k - k_1$, $|\beta| = k_1 - k_2$, $j = 1, \dots, d_2$.

If $d_2 = 1$, then we set $p = 2$ and conclude our construction. Suppose $d_2 > 1$. Analogously as k_2, we can take the smallest entire number k_3 such that $k_3 < k_2$ and

$$(8') \qquad D^{\beta}(D^{\beta^2}(D^{\beta^1} f * \Phi_1) * \Phi_2)(v) = 0$$

for each $|\beta^1| = k - k_1$,
$|\beta^2| = k_1 - k_2$,
$|\beta| \leq k_2 - k_3$,

and for $v = v^0$ (β^1, β^2, β is n-dimensional, d_1-dimensional,

d_2 -dimensional multiindex, respectively). If $k_3 = 0$. then we set . $= 2$. Assume $k_3 > 0$. Then we can (analogously as $Z_{k_1, s}$ Φ_1 , Φ_2) construct the sets Z_{k_1, k_2, k_3} , Z_{k_1, k_2, k_3, k_4} ... and mappings Φ_3 , Φ_4 , ... , respectively. It is easy to see that after a finite number of steps we obtain the following assertion:

Lemma 3.1. To each point $x^0 \in Z$, a finite number of mappings $\Phi_1, \dots, \Phi_p$ and a ball $D(x^0)$ can be associated such that (we use the notation from Construction 3.1)

(9) $\Phi_\ell \in C^{k_\ell \lambda}(D_\ell)$, D_ℓ is a ball in E_{d_ℓ} , $\ell = 1, \dots, p$,

where $k_p < k_{p-1} < \dots < k_1 \leq k$; $d_p < d_{p-1} < \dots < d_1 < n$;

(10) $\Phi_\ell(D_\ell) \subset D_{\ell-1}$, $Z_{k_1, \dots, k_\ell} \cap D(x^0) \subset \Phi_1 * \dots * \Phi_\ell(D_\ell) \subset \Omega$,
$\ell = 1, \dots, p$;

(11) $D^\beta(D^{\beta^\ell}(\dots(D^{\beta^2}(D^{\beta^1} f * \Phi_1) * \Phi_2)\dots) * \Phi_\ell)(v) = 0$

for $v = v^0$, $(\Phi_1 * \dots * \Phi_\ell\ (v^0) = x^0)$,
$|\beta^1| = k - k_1$, $|\beta^2| = k_1 - k_2, \dots, |\beta^\ell| = k_{\ell-1} - k_\ell$,
$|\beta| \leq k_\ell - k_{\ell+1}$ and for $\ell = 1, \dots, p - 1$;

if $d_p > 1$, then this holds for $\ell = p$, $k_{p+1} = 0$, too.

Let us define $\Phi(v) = \Phi_1 * \dots * \Phi_p(v)$ for $v \in D_p$.

Lemma 3.2. There exists a finite number of sets

$Z^1, \dots, Z^q$ such that $\bigcup_{j=1}^{q} Z^j = Z$ and each set Z^j contains all points $x \in Z$ of the same type in the following sense:

if $x^1, x^2 \in Z^j$ and if $\Phi_1^1, \dots, \Phi_{p_1}^1$; $\Phi_1^2, \dots, \Phi_{p_2}^2$; respectively, are the corresponding mappings associated to the points x_1, x_2, respectively, by Lemma 3.1, then $p_1 = p_2$, $k_i^1 = k_i^2$ and the implicit function theorem is used for the same combination of variables in each step of Construction 3.1 (i.e. the domains of Φ_i^1, Φ_i^2 lie in the same subspace of E_m, $i = 1, \dots, p_1 = p_2$).

Proof. The assertion follows from Construction 3.1 and Lemma 3.1.

Remark 3.2. Assume $x^1, x^2 \in Z^j$ (j fixed). Let Φ_i^1, Φ_i^2, $i = 1, \dots, p$ be the corresponding mappings (see Lemma 3.1, 3.2) with the domains D_i^1, D_i^2. Then $\Phi_i^1 = \Phi_i^2$ on $D_i^1 \cap D_i^2$. It follows from the construction of these mappings, from the fact that $x^1, x^2 \in Z^j$ for the same j and from the unicity of the implicit function.

Remark 3.3. Assume $x^0 \in Z^j$. Then the condition (11) is fulfilled for each $v \in D_\ell$ such that $\Phi_1 * \dots * \Phi_\ell(v) \in Z^j$. This follows from Remark 3.2 and from the validity (11) for mappings associated to the point $x = \Phi_1 * \dots \Phi_\ell(v)$.

Remark 3.4. Suppose $x^0 \in Z^j$. Then $D(x^0) \cap Z^j \subset \Phi(D_p)$. This follows from (10), because

$D(x^0) \cap Z^j \subset Z_{k_1,\dots,k_p}$ for some set $Z_{k_1,\dots,k_p}$ (see Construction 3.1 and Remark 3.2).

Proof of Theorem 3.1. An open ball $D(x^0)$ from Lemma 3.1 corresponds to each point $x^0 \in Z^j$. These balls cover Z^j and therefore we can select a countable covering $\{D(x^t)\}_{t=1}^{\infty}$ of the set Z^j. We have a finite number of sets Z^j. Hence, it is sufficient to prove: if $x^0 \in Z^j$ is a fixed point, then there exists a set $M \subset D(x^0) \cap Z^j$ such that $|f(x^1) - f(x^2)| \leq C \|x^1 - x^2\|^{k+\lambda}$ for each $x^1, x^2 \in M$ and the set $Z^j \cap D(x^0) \setminus M$ is countable.

Let $x^0 \in Z^j$ be fixed. We shall use the notation from Construction 3.1 and Lemma 3.1. Denote $A = \{v \in D_p;\ \Phi(v) \in D(x^0) \cap Z^j\}$, $M = \Phi(A' \cap A)$, where A' is the set of all limit points of A. By Remark 3.4, we have $D(x^0) \cap Z^j \subset \Phi(A)$, the set $A \setminus A'$ countable, therefore $D(x^0) \cap Z^j \setminus M$ is countable. Suppose $x^1, x \in M$, $v^1, v \in A'$, $\Phi(v^1) = x^1$, $\Phi(v) = x$. We have $D^{\beta} f(x) = 0$ for $|\beta| \leq k - k_1$ (see Construction 3.1 and Lemma 3.2 - we have $x, x^0 \in Z^j$ for the same j). By Remark 2.2 (we put $F = f$, $\psi = \Phi$)

$$|f(x^1) - f(x)| \leq C \sum_{|\beta| = k - k_1} |D^{\beta} f(\Phi(v^2))| \cdot \|v^1 - v\|^{k-k_1} =$$

(12)

$$= C \sum_{|\beta| = k - k_1} |(D^{\beta} f * \Phi_1)(\Phi_2 * \dots * \Phi_p(v^2))| \cdot \|v^1 - v\|^{k-k_1},$$

where $v^2 \in \overline{v^1 v}$. Lemma 3.1 and Remark 3.3 imply

$$D^{\beta}(D^{\beta^1} f * \Phi_1)(\Phi_2 * \dots \Phi_n(v)) = 0 \text{ for } |\beta^1| = k - k_1 \ ,$$
$$|\beta| \leq k_1 - k_2 \ .$$

From Remark 2.2 we obtain (we put $F = D^{\beta} f * \Phi$, $\psi = \Phi_2 * \dots * \Phi_n$)

$$(13) \quad |(D^{\beta^1} f * \Phi_1)(\Phi_2 * \dots * \Phi_n(v^2))| \leq$$
$$\leq C \sum_{|\beta^2| = k_1 - k_2} |(D^{\beta^2}(D^{\beta^1} f * \Phi_1))(\Phi_2 * \dots * \Phi_n(v^3))| . \|v^2 - v\|^{k_1 - k_2},$$

$\|v^2 - v\| \leq \|v^1 - v\|$. Analogously, we can proceed: we shall estimate $D^{\beta^2}(D^{\beta^1} f * \Phi_1) * \Phi_2 , D^{\beta^3}(D^{\beta^2}(D^{\beta^1} f * \Phi_1) * \Phi_2) * \Phi_3$ etc. After $n - 1$ steps we obtain altogether (from the estimates (12),(13) etc.)

$$(14) \quad |f(x^1) - f(x)| \leq C \sum_{\beta^1, \dots, \beta^n} |D^{\beta^n}(\dots(D^{\beta^2}(D^{\beta^1} f * \Phi_1) * \Phi_2) \dots$$
$$\dots) * \Phi_n(v^{n+1})| . \|v^1 - v\|^{k - k_n} \quad ,$$

the sum is taken over all multiindexes $|\beta^1| = k - k_1, \dots$
$\dots, |\beta^n| = k_{n-1} - k_n$.

If $d_n > 1$, then from Lemma 3.1 and Remark 3.3 it follows

$$D^{\beta}(D^{\beta^n}(\dots(D^{\beta^2}(D^{\beta^1} f * \Phi_1) * \Phi_2) * \dots) * \Phi_n)(v) = 0 \ ,$$
$$|\beta^1| = k - k_1, \dots, |\beta^n| = k_{n-1} - k_n \ , \ |\beta| \leq k_n \ .$$

Hence, we obtain by using (14) and the mean value theorem

$|f(x^1) - f(x)| \leq$

$\leq C \sum_{\beta^1,\dots,\beta^{n+1}} |D^{\beta^{n+1}}(D^{\beta^n}(\dots(D^{\beta^2}(D^{\beta^1}f * \Phi_1) * \Phi_2) * \dots$

$\dots) * \Phi_n)(v^{n+2})| \,.\, \|v^1 - v\|^k \leq$

$\leq C \,.\, \|v^{n+2} - v\|^{\lambda} \,.\, \|v^1 - v\|^k \leq C\|v^1 - v\|^{k+\lambda}$

(the sum being taken over all multiindexes $|\beta^1| = k - k_1, \dots$ $\dots, |\beta^n| = k_{n-1} - k_n, |\beta^{n+1}| = k_n$), because the functions in the middle member are λ-Hölderian.

Suppose $d_n = 1$. The functions which are in the right hand side in (14), are the functions of one variable and they are equal to zero on each point from A (see Remark 3.3). But we have $v \in A'$ and from here we see that the derivatives of all orders not exceeding k_n of these functions on v are equal to zero. Hence, we can conclude the proof analogously as in the case $d_n > 1$.

4. Hausdorff measure of the set of critical values

Theorem 4.1. Let f be a function, $f \in C^1(\Omega)$, $\kappa \geq 1$. Let A be a compact subset of Z and

(15) $|f(x') - f(x)| \leq C \,.\, \|x' - x\|^{\kappa}$

for each $x', x \in A$, where $C > 0$. Then $f(A)$ is $\frac{n}{\kappa}$-null.

Proof. For each positive integer N we shall denote by $\{I_N^j\}_{j=1}^{\kappa_N}$ a system of all intervals of the type

$\langle k_1 N^{-1}, (k_1+1)N^{-1}\rangle \times \ldots \times \langle k_n N^{-1}, (k_n+1)N^{-1}\rangle$

(n -dimensional cubes) which intersect the set A (k_i are entire numbers). Set $J_N^j = I_N^j \cap A$. We have $\bigcup_j J_N^j = A$, therefore $\bigcup_j f(J_N^j) = f(A)$. From (15) we obtain $diam\, f(J_N^j) \leq C \cdot N^{-\kappa}$. By the definition of Hausdorff measure we have

$$(16) \qquad \mu_{\frac{n}{\kappa}}(f(A)) \leq \lim_{N\to\infty} \sum_{j=1}^{\kappa_N} [diam\, f(J_N^j)]^{\frac{n}{\kappa}} .$$

Let $\varepsilon > 0$ be arbitrary (but fixed). Let us divide the sets J_N^j for each fixed N into two groups:

(i) $\quad diam\, f(J_N^j) \leq \varepsilon N^{-\kappa}$;

(ii) $\quad diam\, f(J_N^j) > \varepsilon N^{-\kappa}$.

By $\nu_N^{(1)}$, $\nu_N^{(2)}$ respectively, denote the number of sets which lie in the group (i),(ii). Put $\nu_N = \nu_N^{(1)} + \nu_N^{(2)}$. Let us suppose that we have proved the following assertion:

$$(17) \qquad \nu_N = O(N^n) , \quad \nu_N^{(2)} = o(N^n) .$$

Then

$$\sum_{j=1}^{\kappa_N} [diam\, f(J_N^j)]^{\frac{n}{\kappa}} = \sum_{J_N^j \in (i)} [diam\, f(J_N^j)]^{\frac{n}{\kappa}} +$$
$$+ \sum_{J_N^j \in (ii)} [diam\, f(J_N^j)]^{\frac{n}{\kappa}} \leq \nu_N^{(1)}(\varepsilon N^{-\kappa})^{\frac{n}{\kappa}} + \nu_N^{(2)}(C_1 N^{-\kappa})^{\frac{n}{\kappa}} \leq$$
$$\leq \varepsilon^{\frac{n}{\kappa}} \nu_N^{(1)} N^{-n} + C_2 \nu_N^{(2)} N^{-n} .$$

The second member in the right hand side converges to zero (if $N \to \infty$) by (17) and the first member can be made arbitrarily small by a convenient choice of ε. From here and from (16) we obtain $f(A)$ is $\frac{n}{k}$ -null.

Hence, it is sufficient to prove (17).

Suppose

(18) there exists $\delta > 0$ (dependent of ε only, independent of N, j) such that $m_n(J_N^j) \leq (1-\delta) N^{-n}$ for each $J_N^j \in (ii)$ (where m_n denotes the n-dimensional Lebesgue measure).

Set $A_N = \nu_N N^{-n} - m_n(A)$. We have $A_N \to 0$, because A is compact. From here $\nu_N = O(N^n)$. We have

$$m_n(A) \leq \nu_N^{(1)} N^{-n} + (1-\delta)\nu_N^{(2)} N^{-n},$$

hence

$$\nu_N^{(1)} + \nu_N^{(2)} = m_n(A) N^n + o(N^n) \leq \nu_N^{(1)} + (1-\delta)\nu_N^{(2)} + o(N^n).$$

From here $\delta \nu_N^{(2)} = o(N^n)$, i.e. $\nu_N^{(2)} = o(N^n)$, hence (17) is valid. Hence, it is sufficient to prove (18).

Let J_N^j be an arbitrary set of the group (ii). There exist $a, b \in J_N^j$ such that $\operatorname{diam} f(J_N^j) = f(b) - f(a) > \varepsilon N^{-k}$.

From (15) we obtain

(19) $$|f(b') - f(a')| \geq \frac{1}{2} \varepsilon N^{-k}$$

for each

$$(20)\quad a', b' \in I_N^j, \; \|a'-a\| < \left(\frac{\varepsilon}{4C}\right)^{\frac{1}{\kappa}} N^{-1}, \; \|b'-b\| < \left(\frac{\varepsilon}{4C}\right)^{\frac{1}{\kappa}} N^{-1} .$$

Consider two points a', b' which fulfil (20) and $\overline{a'b'} \cap A \neq \emptyset$. Then there exist the open segments S_i , $i = 1, 2, \ldots$ such that $\overline{a'b'} \setminus A = \bigcup_{i=1}^{\infty} S_i$. Denote the extreme points of these segments by a^i, b^i. We obtain

$$|f(b') - f(a')| \leq \sum_{i=1}^{\infty} |f(b^i) - f(a^i)| \leq$$
$$\leq C \cdot \sum_{i=1}^{\infty} (diam\, S_i)^{\kappa} \leq C \cdot \left(\sum_{i=1}^{\infty} diam\, S_i\right)^{\kappa} =$$
$$= C \cdot [m_1(\overline{a'b'} \setminus A)]^{\kappa} .$$

If $m_1(\overline{a'b'} \setminus A) < \left(\frac{\varepsilon}{2C}\right)^{\frac{1}{\kappa}} N^{-1}$, then we obtain $|f(b') - f(a')| < \frac{1}{2} \varepsilon N^{-\kappa}$. But it is not possible by (19),(20), hence

$$(21)\ \text{if}\ \|a'-a\| \leq \frac{1}{4}\left(\frac{\varepsilon}{C}\right)^{\frac{1}{\kappa}} N^{-1}, \; \|b'-b\| \leq \frac{1}{4}\left(\frac{\varepsilon}{C}\right)^{\frac{1}{\kappa}} N^{-1} ,$$
$$\overline{a'b'} \setminus A \neq 0, \text{ then } m_1(\overline{a'b'} \setminus A) \geq \frac{1}{2}\left(\frac{\varepsilon}{C}\right)^{\frac{1}{\kappa}} N^{-1} .$$

If $\overline{a'b'} \cap A = \emptyset$, then the last inequality holds, too. It is easy to see there exists $C_4 > 0$ (dependent of the dimension n only, independent of j, N) such that there exist $a^0, b^0 \in I_N^j$ which fulfil the conditions

$$D(a^0, C_4\, \varepsilon^{\frac{1}{\kappa}} N^{-1}) \subset D(a, \tfrac{1}{4}(\tfrac{\varepsilon}{C})^{\frac{1}{\kappa}} N^{-1}) \cap I_N^j \;;$$

$$D(b^o, C_4 \varepsilon^{\frac{1}{k}} N^{-1}) \subset D(b, \frac{1}{4}(\frac{\varepsilon}{C})^{\frac{1}{k}} N^{-1}) \cap I_N^{z} .$$

Let K be a convex closure of the set $D(a^o, C_4 \varepsilon^{\frac{1}{k}} N^{-1}) \cup D(b^o, C_4 \varepsilon^{\frac{1}{k}} N^{-1})$.
By using (21) we obtain

$$m_n(K \setminus A) \geq P \frac{1}{2} \left(\frac{\varepsilon}{C}\right)^{\frac{1}{k}} N^{-1} ,$$

where P is the volume of $(n-1)$-dimensional ball with $diam\ P = 2 . C_4 \varepsilon^{\frac{1}{k}} N^{-1}$. It is easy to see from here

$$m_n(K \setminus A) \geq C_5 \varepsilon^{\frac{n}{k}} N^{-n} ,$$

where C_5 depends on C and n only. Further, $m_n(I_N^{z} \setminus A) \geq m_n(K \setminus A)$.

It is sufficient to put $\delta = C_5 \varepsilon^{\frac{n}{k}}$ and the assertion (18) is proved. This completes the proof of Theorem 4.1.

Theorem 4.2. If $f \in C^{k,\lambda}(\Omega)$ is a function, then the set $f(Z)$ is $\frac{n}{k+\lambda}$-null.

Proof. It is easy to see that we can suppose that the sets M_t from Theorem 3.1 are compact. Our assertion follows from here and from Theorem 4.1.

Remark 4.1. If $s < \frac{n}{k+\lambda}$, then there exists a function from the class $C^{k,\lambda}$ such that $\mu_s(f(Z)) > 0$ (see [1]).

Remark 4.2. If $f \in C^{\infty}$ (i.e. f has continuous derivatives of all orders), then the set $f(Z)$ is s-null for each $s > 0$. This follows from Theorem 4.2. But the set $f(Z)$ need not be countable. We must demand f is real-analytic to obtain such a strong assertion (see [5]).

R e f e r e n c e s

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