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COEQUALIZERS IN THE GENERALIZED ALGEBRAIC CATEGORIES

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The present paper is devoted to the study of the existence of coequalizers in a certain class of categories which is a generalization of the categories of universal algebras of a given type. This class includes also categories of topological spaces, categories of convergent spaces and the like (see [3],[6]). A generalized algebraic category is a category  $A(F, G, \sigma)$  where  $F, G$  are set functors (i.e. functors from the category  $\mathcal{S}$  of sets into itself),  $\sigma = \{\alpha_i\}_{i \in I}$  is a type ( $\alpha_i$  ordinals). Its objects are pairs  $\langle X, \{\omega_i\}_{i \in I} \rangle$  where  $\omega_i : (FX)^{\alpha_i} \rightarrow GX$  and its morphisms from  $\langle X, \{\omega_i\} \rangle$  to  $\langle X', \{\omega'_i\} \rangle$  are mappings  $f: X \rightarrow X'$  such that for every  $i \in I$  the diagram consisting of  $\omega_i, Gf, \omega'_i, (Ff)^{\alpha_i}$  is commutative. Our concern will be the covariant case (both  $F$  and  $G$  covariant) and the contravariant case (both functors contravariant).

Several papers are devoted to the study of limits and colimits in the generalized algebraic categories. In the present one, we give a necessary and sufficient condition

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for the existence of coequalizers in the covariant and contravariant cases as well as the condition for their preservation by the natural forgetful functor (i.e. the functor assigning to every  $\langle X, \{\omega_i\} \rangle$  its underlying set  $X$  and to every morphism its underlying mapping).

The paper has two parts: the first one contains preliminaries, the second one brings the main results.

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### 1. Preliminaries

Note 1.1. We say "functor" instead of covariant set functor, while "set functor" indicates that both variances are considered. We consider set functors up to natural equivalence.

We proved that every functor is naturally equivalent with an inclusions-preserving one (i.e. if  $\emptyset \neq A \subset B$ , then  $FA \subset FB$  and for  $i: A \rightarrow B$ ,  $i(x) = x$  we have  $F(i(\eta)) = \eta$ ) - see [7]. Thus we assume that all functors throughout this paper preserve inclusions. In particular, for arbitrary  $f: X \rightarrow Y$ ,  $A \subset X$  we have  $F(f/A) = Ff/FA$ ,  $im Ff = Fimf$ .

Note 1.2. Let us present some functors:  $\mathcal{Q}_M$  - the cartesian power,  $C_{M, \mu, N}$  - the constant functor ( $\mu: M \rightarrow N$ ) which is constant to  $\mu$  on the category  $\mathcal{S}'$  of non-void sets,  $C_{M, \mu, N} \emptyset = M$ ,  $C_{M, \mu, N} f = \mu$ ,

if  $\text{dom } f = \emptyset$ . Put  $C_N = C_{N, id_N, N}$ . Contravariant functors:  $P_M$  - the contravariant homfunctor,  $P_{M, \mathcal{F}}$  - its factorfunctor ( $\mathcal{F} \subset M^M$  such that  $\sim_X$  is an equivalence on  $M^X$  for every  $X$ , where  $\alpha \sim_X \beta$  iff there exist  $\varphi \in \mathcal{F}$  with  $\varphi \alpha = \varphi \beta$ ):  $P_{M, \mathcal{F}} X = P_M X / \sim_X$ .  $C_{0, M}^*$  - on  $\mathcal{S}$  const  $\emptyset$ ,  $C_{0, M}^* \emptyset = M$ .

Note 1.3. We call a functor  $F$  connected if  $|F1| = 1$ .

Denote  $\bigvee_{i \in I} F_i$  the disjoint union of  $\{F_i\}_{i \in I}$ .  $F$  is connected iff whenever  $F = F_1 \vee F_2$  then either  $F_1 = C_{\emptyset}$  or  $F_2 = C_{\emptyset}$ .

If  $F$  is a contravariant functor with  $F \neq C_{0, M}^*$ , then always  $FX \neq \emptyset$ . If  $f$  is an epimorphism, then  $Ff$  is an epimorphism if  $F$  is covariant,  $Ff$  is a monomorphism if  $F$  is contravariant.

Definition 1.4. A mapping  $f$  is coarser than a mapping  $g$  iff  $\text{dom } f = \text{dom } g$  and  $g(x) = g(y) \implies f(x) = f(y)$  (or, equivalently, iff there exists  $\tau$  with  $f = \tau \circ g$ ).

Let  $\{f_i\}_{i \in I}$  be a non-void collection of mappings with common domain  $X \neq \emptyset$ .

Co-join of  $\{f_i\}$  is such an epimorphism  $f = \bigcup_{i \in I}^* f_i$  that

1) all  $f_i$  are coarser than  $f$  and 2) if all  $f_i$  are coarser than a mapping  $g$ , then also  $f$  is coarser than  $g$ . If  $\bigcup_{i \in I}^* f_i = id_X$ , we call  $\langle X, \{f_i\}_{i \in I} \rangle$  a separating system.

Definition 1.5. A functor  $F$  preserves coequalizers (resp. separating systems), if for every  $f, g: X \rightarrow Y$

with a coequalizer  $\kappa$ ,  $F\kappa$  is the coequalizer of  $Ff$ ,  $Fg$  (resp. for every separating system  $\langle X, \{f_i\}_{i \in I} \rangle$ , also  $\langle FX, \{Ff_i\}_{i \in I} \rangle$  is separating). A contravariant functor  $F$  turns coequalizers into equalizers if for every  $f, g: X \rightarrow Y$  with a coequalizer  $\kappa$ ,  $F\kappa$  is the equalizer of  $Ff, Fg$ .

Convention 1.6. A type is 0-unary if all its elements are 0 or 1.

Let  $f$  be a mapping.  $f \equiv a$  denotes that  $f$  is constant to  $a$ .

Note 1.7. A functor  $F$  preserves coequalizers iff it preserves countable unions (i.e. always  $F(\bigcup_{m=1}^{\infty} X_m) = \bigcup_{m=1}^{\infty} FX_m$ ). See [5]. A contravariant functor turns coequalizers into equalizers iff it has the form  $\bigcup_{i \in I} P_{M_i}, \mathcal{F}_i$  - more precisely see [7].

Note 1.8. It is proved in [7] that the following properties of a functor  $F$  are equivalent:

- 1)  $F$  is connected and it preserves separating systems,
- 2)  $F$  is connected and it preserves co-joins (i.e. always  $F(\bigcup^* f_i) = \bigcup^* Ff_i$ ),
- 3) for every set  $X$  and every  $x, y \in FX$  there exists a mapping  $\kappa$  with  $\text{dom } \kappa = X$  and  $F\kappa(x) = F\kappa(y)$  such that every mapping with the same property is coarser than  $\kappa$ .

Note 1.9. Denote  $A(F, G, 1) = A(F, G, \{1\})$ . It is easy to verify that  $A(F, G, \mathcal{G})$  with  $\mathcal{G} = \{M_i\}_{i \in I}$  is isomorphic to  $A((\bigvee_{i \in I} G_{M_i}), F, G, 1)$ .

Clearly, for an arbitrary functor  $F, \bigvee_{i \in I} \mathcal{O}_{M_i} \cdot F$  preserves coequalizers iff  $F$  preserves them and  $|M_i| \leq 1$  or  $F$  is constant,  $M_i$  arbitrary.

**Note 1.10.** Let  $\mathcal{K}$  be the coequalizer of  $\varphi, \psi: M \rightarrow N$  in  $\mathcal{S}$ . Then  $\mathcal{K}^{-1}\mathcal{K}(A) = A \Rightarrow \varphi^{-1}(A) = \psi^{-1}(A)$ .

**Proof.**  $t \in \varphi^{-1}(A), \mathcal{K}\varphi = \mathcal{K}\psi \Rightarrow \psi(t) \in \mathcal{K}^{-1}\mathcal{K}(A) = A \Rightarrow t \in \psi^{-1}(A)$ .

Analogously,  $t \in \psi^{-1}(A) \Rightarrow t \in \varphi^{-1}(A)$ .

**Proposition 1.11.** Let  $F$  be a functor not preserving coequalizers. Then for every set  $Z$  there exist mappings  $f, g: X \rightarrow Y$  such that  $Z \subset X \cap Y, f/Z = g/Z$  is the inclusion, and there exist disjoint  $A, B \subset FY$  such that if we denote  $\mathcal{K}$  the coequalizer of  $f$  and  $g, \mathcal{K}^*$  an epimorphism from  $Y, \mathcal{K}^*/Z$  -constant,  $\mathcal{K}^*/Y-Z \cong \mathcal{K}/Y-Z$ , we have

- 1)  $F\mathcal{K}A \cap F\mathcal{K}B \neq \emptyset$ ,
- 2)  $x \in A \cup B, F\mathcal{K}^*(x) = F\mathcal{K}^*(y) \Rightarrow y \in A \cup B$ ,
- 3)  $(Ff)^{-1}A = (Fg)^{-1}A, (Ff)^{-1}B = (Fg)^{-1}B$ .

To prove this proposition we shall need some special facts concerning set functors, which we present separately.

**Definition A.** Define a functor  $\underline{F}; FX = \{ \tilde{f}, \tilde{f} = \text{exp } X \text{ or } \tilde{f} \text{ is a filter on } X \}$ , for  $f: X \rightarrow Y$ ,  $Ff(\tilde{f}) = \{ B; B \subset Y, f(A) \subset B \text{ for some } A \in \tilde{f} \}$ . Let  $F$  be a functor. Define  $\mathcal{F}_F^X: FX \rightarrow FX$  for

every set  $X : \mathcal{F}_F^X(x) = \{A; A \subset X, x \in FA\}$ .

Statement B. Let  $F$  be a functor,  $f: X \rightarrow Y$  a monomorphism,  $x \in FX$ . Then  $\mathcal{F}_F^Y(Ff(x)) = Ff(\mathcal{F}_F^X(x))$ .

Proof. See [1].

Statement C. A functor  $F$  preserves coequalizers iff for every  $x \in FX$ ,  $\mathcal{F}_F^X(x)$  is an ultrafilter closed under countable intersections. or  $\text{exp } X$ .

Proof. See [5].

Statement D. A filter  $\tilde{t}$  on  $X$  is an ultrafilter closed under countable intersections iff for every countable disjoint decomposition of  $X$ ,  $X = \bigcup_{i=1}^{\infty} X_i$ , there exists  $m$  with  $X_m \in \tilde{t}$ .

Proof of Proposition 1.11. If  $F$  does not preserve coequalizers, there exists  $\xi \in FY$  such that  $\mathcal{F}_F^Y(\xi)$  is neither  $\text{exp } Y$  nor an ultrafilter closed under countable intersection and so there exists a disjoint decomposition of  $Y$ ,  $Y = \bigcup_{i=1}^{\infty} Y_i$  with  $Y_m \notin \mathcal{F}_F^Y(\xi)$  for every  $m = 1, 2, \dots$ . Let  $\mathbb{Z}$  be the set of all integers and put  $X_1 = \mathbb{Z} \times Y$ . Without loss of generality  $X_1 \cap \mathbb{Z} = \emptyset$ . Put  $X = X_1 \cup \mathbb{Z}$ . Put  $f, g: X \rightarrow X$ ,  $f = id_X$ ,  $g/\mathbb{Z} = id/\mathbb{Z}$ ,  $g(\langle k, y \rangle) = \langle k+1, y \rangle$ . Let  $l, h: Y \rightarrow X$ ,  $l(y) = \langle 0, y \rangle$ ,  $h(y) = \langle m, y \rangle \iff y \in Y_m$ . Notice that  $f, g, l, h$  are monomorphisms and thus they fulfil the assumptions of the statement B. Further,  $g$  is an automorphism and so  $g^k, k \in \mathbb{Z}$  has sense. As  $Ff = id$  and  $Fg$  is an automorphism,  $X(t) = X(u) \iff \exists m \in \mathbb{Z}, t = (Fg)^m(u)$ ,

where  $K$  is the coequalizer of  $Ff$  and  $Fg$ . Let us prove:  $K(Fh(\xi)) \neq K(Fl(\xi))$ . Assume that, on the contrary,  $Fh(\xi) = (Fg)^m(Fl(\xi))$  for some  $m \in \mathbb{Z}$ . Then

$$\begin{aligned} q^m l(Y) \in F(q^m l)(\mathcal{F}_F^Y(\xi)) &= \mathcal{F}_F^X(Fq^m l(\xi)) = \\ &= \mathcal{F}_F^X(Fh(\xi)) = Fh(\mathcal{F}_F^Y(\xi)) \end{aligned}$$

and so there exists  $H \in \mathcal{F}_F^Y(\xi)$  with  $h(H) \subset q^m l(Y)$ . But then  $H \subset Y_m$  (see the definition of  $h, l$  and  $q$ ) and that is a contradiction as  $H \in \mathcal{F}_F^Y(\xi)$ ,  $Y_m \supset H \Rightarrow Y_m \in \mathcal{F}_F^Y(\xi)$ . Therefore  $K(Fl(\xi)) \neq K(Fh(\xi))$ .

Let  $h, h^*$  be as in the proposition. Evidently,  $h^*/X_1$  is the projection  $X \times Y \rightarrow Y$  and so  $h^* l = h^* h$  and  $Fh^*(Fl(\xi)) = Fh^*(Fh(\xi))$ . Now put  $A = K^{-1}K(Fl(\xi))$ ,  $B = (Fh^*)^{-1}(Fh^*(Fl(\xi))) - A$ . Evidently,  $A \cap B = \emptyset$ .

- 1)  $Fl(\xi) \in A$ ,  $Fh(\xi) \in B$  and  $Fh^*(Fl(\xi)) = Fh^*(Fh(\xi))$ ,
- 2)  $x \in A \cup B$ ,  $Fh^*(x) = Fh^*(y) \Rightarrow y \in A \cup B$ ,
- 3) as  $Fh^*Ff = Fh^*Fg$ ,  $Fh^*$  is coarser than  $K$  and so  $K^{-1}(K(B)) = B$ . Then,  $(Ff)^{-1}A = (Fg)^{-1}A$ ,  $(Ff)^{-1}B = (Fg)^{-1}B$  (see 1.10).

**Lemma 1.12.** Let  $A(F, C_{M \times N}, 1)$  have coequalizers. Then either  $F$  preserves coequalizers or  $|N| \leq 1$ .

**Proof.** Let neither hold. Then there exist  $f, g: X \rightarrow Y$  with a coequalizer  $h$  such that there is  $A \subset FY$  with



$(Ff)^{-1}A = (Fg)^{-1}A = \emptyset$  and there exist  $a \in A$ ,  
 $b \in FY - A$ ,  $Fh(a) = Fh(b)$  (see Lemma 2.2).  
 (Necessarily  $X \neq \emptyset \neq Y$ .) Let  $t, u \in N$ ,  $t \neq u$ ,  
 let  $\omega^X: FX \rightarrow N$ ,  $\omega^X/A \cong t$ ,  $\omega^X/FX - A \cong u$ ; let  
 $\omega^Y: FY \rightarrow N$ ,  $\omega^Y/A \cong t$ ,  $\omega^Y/FY - A \cong u$ . Obvious-  
 ly  $f, g: \langle X, \omega^X \rangle \rightarrow \langle Y, \omega^Y \rangle$  and this couple does  
 not have a coequalizer in  $A(F, C_{M \times N}, 1)$ . Indeed,  
 let  $l: \langle Y, \omega^Y \rangle \rightarrow \langle V, \omega^V \rangle$  and  $lf = lg$ . Then  $l$   
 is coarser than  $h$  and so  $Fh(a) = Fh(b)$ . But  
 $\omega^V Fh(a) = \omega^V(a) = t \neq u = \omega^V(b) = \omega^V Fh(b)$  which is a  
 contradiction.

## 2. Main results.

The covariant case. Theorem 2.1.  $A(F, G, \sigma)$  with  
 $F, G$  covariant has coequalizers preserved by the forget-  
 ful functor iff  
 either  $F$  preserves coequalizers and  $\sigma$  is 0-ary,  
 or  $F$  is constant,  
 or  $G = C_{M \times N}$  with  $|N| \leq 1$ .

Proof. I. Let  $A(F, G, \sigma)$  have such coequalizers.  
 With the help of 1.9 it suffices to prove that if, moreover,  
 $\sigma = \{1\}$  and  $G \neq C_{M \times 1}, C_0$ , then  $F$  preserves  
 coequalizers. Assume the contrary. Evidently there exists  
 $Z \neq \emptyset$  with  $|GZ| > 1$ , let  $a, b \in GZ$ ,  $a \neq b$ .  
 Let us use the proposition 1.11 to  $F, Z$ . Thus we have

$f, g: X \rightarrow Y$  with  $Z \subset X \cap Y$ ,  $f|_Z = g|_Z = id|_Z$   
 and  $A \subset FY$ ,  $Fk(A) \cap Fk(FY - A) = \emptyset$  (where  $k$   
 is the coequalizer of  $f$  and  $g$  in  $\mathcal{S}$ ),  $(Ff)^{-1}A =$   
 $= (Fg)^{-1}A = \{a\}$ . Let  $\alpha \in A$ ,  $\beta \in FY - A$ ,  
 $Fk(\alpha) = Fk(\beta)$ . Put  $\omega^X: FX \rightarrow GX$ ,  $\omega^X/a \equiv a$ ,  
 $\omega^X/FX - a \equiv \emptyset$ ,  $\omega^Y: FY \rightarrow GY$ ,  $\omega^Y/A \equiv Gf(a)$ ,  $\omega^Y/FY - A \equiv Gf(b)$ .

Evidently  $f, g: \langle X, \omega^X \rangle \rightarrow \langle Y, \omega^Y \rangle$ . Let  
 $k: \langle Y, \omega^Y \rangle \rightarrow \langle V, \omega^V \rangle$  be a coequalizer of  $f$  and  $g$   
 in  $A(F, G, 1)$ . Clearly,  $k|_Z$  is a monomorphism  
 and so  $Gk|_{GZ}$  is a monomorphism,  $Gk(Gf(a)) \neq$   
 $\neq Gk(Gf(b))$ . But  $Gk(Gf(a)) = Gk(\omega^Y(\alpha)) =$   
 $= \omega^V Fk(\alpha) = \omega^V Fk(\beta) = Gk \omega^Y(\beta) = Gk(Gf(b))$   
 which is a contradiction.

II. Evidently  $A(F, C_{M \times N}, \sigma)$  with  $|N| \leq 1$   
 has coequalizers preserved by the forgetful functor. Now  
 it suffices to prove that if  $F$  preserves coequalizers,  
 $A(F, G, 1)$  has such coequalizers. Let  $f, g: \langle X, \omega^X \rangle \rightarrow$   
 $\rightarrow \langle Y, \omega^Y \rangle$ , let  $k: Y \rightarrow Z$  be a coequalizer of  $f$   
 and  $g$  in  $\mathcal{S}$ . As  $Fk$  is the coequalizer of  $Ff$  and  
 $Fg$  and as  $(Gk \cdot \omega^Y) \cdot Ff = Gk \cdot Gf \cdot \omega^X =$   
 $= Gk \cdot Gg \cdot \omega^X = (Gk \omega^Y) \cdot Fg$ , there exists  $\omega$  with  
 $Gk \cdot \omega^Y = \omega \cdot Fk$ . Thus  $k: \langle Y, \omega^Y \rangle \rightarrow \langle Z, \omega \rangle$ .  
 Let us prove that this is the coequalizer in  $A(F, G, 1)$ .

Let  $l: \langle Y, \omega' \rangle \rightarrow \langle Y, \omega^V \rangle$  be arbitrary with  $lf = lq$ . There exists a unique  $\tau$  with  $l = \tau k$ . Let us show that  $\tau: \langle Z, \omega \rangle \rightarrow \langle Y, \omega^V \rangle$ . We have  $\omega^V \cdot F\tau \cdot Fk = \omega^V \cdot Fl = Gl \cdot \omega^V = G\tau \cdot Gk \cdot \omega^V = G\tau \cdot \omega \cdot Fk$  and as  $Fk$  is an epimorphism, there follows  $\omega^V \cdot F\tau = G\tau \cdot \omega$ .

**Theorem 2.2.**  $A(F, G, \sigma)$ , with  $F$  and  $G$  covariant, has coequalizers iff either  $F$  preserves coequalizers and  $\sigma$  is 0-unary, or  $F$  is constant, or  $G$  is connected and it preserves separating systems.

**Proof.** Assume  $\sigma = \{1\}$ .

I) Let  $A(F, G, 1)$  have coequalizers. Let  $F$  not preserve coequalizers. We shall show that  $G$  has the property 3) from 1.8. We assume that  $G$  is not constant because otherwise (due to 1.12) it is connected, preserving separating systems.

a) To prove 3) we shall, for arbitrary  $Z; x_1, x_2 \in GZ$ , construct  $f, g: \langle X, \omega^X \rangle \rightarrow \langle Y, \omega^Y \rangle$  with  $Z \subset Y$  and such that: if  $\varphi: Z \rightarrow T$ , then

(\*)  $G\varphi(x_1) = G\varphi(x_2)$  iff there exists

$\tilde{\varphi}: \langle Y, \omega^Y \rangle \rightarrow \langle T', \omega^{T'} \rangle$  such that  $\tilde{\varphi}|_Z = \varphi$  and

$\tilde{\varphi}f = \tilde{\varphi}g$ . That will be sufficient because then if

$l: \langle Y, \omega^Y \rangle \rightarrow \langle Y, \omega \rangle$  is the coequalizer of  $f$  and  $g$  in  $A(F, G, 1)$ , then  $\kappa$  - the domain-range restriction of  $l$  to  $Z$  and  $l(Z)$  is the mapping from 3):

first  $G\kappa(x_1) = G\kappa(x_2)$  (as there exists  $\tilde{\kappa} = l$ ), second, if  $G\kappa'(x_1) = G\kappa'(x_2)$ , then there exists  $\tilde{\kappa}'$  and as  $l$  is the coequalizer, there exists  $\tau$  with  $\tilde{\kappa}' = \tau l$  and so  $\kappa' = [\tau/\kappa(x)] \cdot \kappa \cdot b$ . Now let us construct  $f, g$ . As  $G$  is not constant, there exists  $Z', GZ \not\subseteq GZ'$ ; let  $z \in GZ' - GZ$ . Let us use the proposition 1.11 on  $F, Z'$ . We have  $f, g: X \rightarrow Y; A, B \subset Y$  such that  $(\kappa, \kappa^*$  as in 1.11)

$Z' \subset X \cap Y, f/Z' = g/Z'$  is the inclusion,

$A \cap B = \emptyset, F\kappa(A) \cap F\kappa(B) \neq \emptyset, (F\kappa^*)^{-1}F\kappa^*A = A,$

$(F\kappa^*)^{-1}F\kappa^*B = B, (Ff)^{-1}A = (Fg)^{-1}A = A, (Ff)^{-1}B = (Fg)^{-1}B = B.$

Let  $x'_1, x'_2, z' \in GZ', Gf(x'_i) = Gg(x'_i) = x_i, i = 1, 2; Gf(z') = Gg(z') = z.$

Let us define

$\omega^x: FX \rightarrow GX, \omega^x/A \equiv x'_1, \omega^x/B \equiv x'_2, \omega^x/FX-(A \cup B) \equiv z';$

$\omega^y: FY \rightarrow GY, \omega^y/A \equiv x_1, \omega^y/B \equiv x_2, \omega^y/FY-(A \cup B) \equiv z.$

Clearly  $f, g: \langle X, \omega^x \rangle \rightarrow \langle Y, \omega^y \rangle$ . Let us prove (\*).

1) Let  $\varphi: Z \rightarrow T, G\varphi(x_1) = G\varphi(x_2)$ . Let

$\tilde{\varphi}: Y \xrightarrow{\text{onto}} T, \tilde{\varphi}/Z = \varphi, \tilde{\varphi}/Y-Z = \kappa/Y-Z$ . Let us

show that  $G\tilde{\varphi} \circ \omega^y$  is coarser than  $F\tilde{\varphi}$ . Clearly,  $\kappa^*$  is coarser than  $\tilde{\varphi}$ . Therefore  $F\tilde{\varphi}(t) = F\tilde{\varphi}(u) \implies$

$\implies F\kappa^*(t) = F\kappa^*(u) \implies$  either both  $t$  and  $u$  are elements of  $A \cup B$  or neither of them  $\implies$  either

$\omega(t), \omega(u) \in GZ$  and so  $G\varphi \circ \omega(t) = G\varphi \circ \omega(u)$

or  $\omega(t) = \omega(u)$ . Therefore  $G\tilde{\varphi} \circ \omega^y$  is really

coarser than  $F\tilde{\varphi}$  and we may define  $\omega^{\tilde{T}}$  by  
 $\omega^{\tilde{T}}(F\tilde{\varphi}(t)) = G\tilde{\varphi}(\omega(t))$ . Clearly  $\tilde{\varphi}: \langle X, \omega^Y \rangle \rightarrow$   
 $\rightarrow \langle \tilde{T}, \omega^{\tilde{T}} \rangle$ ,  $\tilde{\varphi}f = \tilde{\varphi}g$ .

2) Let  $\tilde{\varphi}: \langle Y, \omega^Y \rangle \rightarrow \langle T_1, \omega^{T_1} \rangle$ ,  $\tilde{\varphi}f = \tilde{\varphi}g$ . Then  $\tilde{\varphi}$   
is coarser than  $k$  and as  $Fk(a) = Fk(b)$  for some  
 $a \in A$ ,  $b \in B$ , we have  $F\tilde{\varphi}(a) = F\tilde{\varphi}(b)$ . Therefore  
 $G\tilde{\varphi}(x_1) = G\tilde{\varphi}(\omega^Y(a)) = \omega^{\tilde{T}}F\tilde{\varphi}(a) = \omega^{\tilde{T}}F\tilde{\varphi}(b) = G\tilde{\varphi}(\omega^Y(b)) = G\tilde{\varphi}(x_2)$ .  
Therefore if  $\varphi = \tilde{\varphi}/Z$ , then  $G\varphi(x_1) = G\varphi(x_2)$ .

II) Due to Theorem 2.1 it is sufficient to show that if  $G$   
preserves separating systems and it is connected, then  
 $A(F, G, 1)$  has coequalizers:

Let  $f, g: \langle X, \omega^X \rangle \rightarrow \langle Y, \omega^Y \rangle$ , let  $\mathcal{M} =$   
 $= \{ \beta; \beta: Y \rightarrow Y', \beta f = \beta g \text{ and } G\beta \circ \omega^Y \text{ coarser than } G\beta \}$ ,  $\mathcal{M} \neq \emptyset$ , as  $\beta \in \mathcal{M}$  if  $\beta$  is constant  
(as then  $G\beta$  is constant while  $G$  is connected). Let  
 $l = \cup^* \mathcal{M}$ ,  $l: Y \rightarrow Y'$ , then  $G_l = \cup_{\beta \in \mathcal{M}}^* G\beta$  (see  
1.8). Therefore  $F_l(t) = F_l(\mu) \implies F\beta(t) = F\beta(\mu)$   
for each  $\beta \in \mathcal{M} \implies G\beta \omega^Y(t) = G\beta \omega^Y(\mu)$  for each  
 $\beta \in \mathcal{M} \implies G_l \omega^Y(t) = G_l \omega^Y(\mu)$  and we may define  $\omega^V:$   
 $FV \rightarrow GV$  by  $\omega^V(F_l(t)) = G_l(\omega^Y(t))$ . Let us  
show that  $l: \langle Y, \omega^Y \rangle \rightarrow \langle V, \omega^V \rangle$  is the coequali-  
zer of  $f$  and  $g$  in  $A(F, G, 1)$ . If  $l': \langle Y, \omega^Y \rangle \rightarrow$   
 $\rightarrow \langle V', \omega^{V'} \rangle$ ,  $l'f = l'g$ , then  $G_l' \circ \omega^Y (= \omega^{V'} F_l')$  is  
coarser than  $F_l'$  and so  $l' \in \mathcal{M}$ . Therefore there

exists a unique  $\tau$  with  $l' = \tau l$ . Let us prove that

$$\tau : \langle V, \omega^V \rangle \longrightarrow \langle V', \omega^{V'} \rangle .$$

$$\omega^{V'} \cdot F\tau \cdot Fl = \omega^{V'} \cdot Fl' = Gl' \cdot \omega^Y = G\tau \cdot Gl \cdot \omega^Y = G\tau \cdot \omega^V Fl$$

and as  $l$  (and  $Fl$ ) is an epimorphism, we have

$$\omega^{V'} \cdot F\tau = G\tau \cdot \omega^V$$

The contravariant case. Theorem 2.3. Let  $F, G$  be contravariant functors. The following statements are equivalent:

- 1)  $A(F, G, \sigma)$  has coequalizers,
- 2)  $A(F, G, \sigma)$  has coequalizers preserved by the forgetful functor,
- 3) either  $G$  turns coequalizers into equalizers, or  $F = C_{0,M}^*$ .

Proof. Evidently it suffices to prove the theorem for  $\sigma = \{1\}$ .  $1) \implies 3)$ . Let  $F \neq C_{0,M}^*$  and let  $G$  not turn coequalizers into equalizers. Let  $f, g: X \rightarrow Y$  with a coequalizer  $k: Y \rightarrow Z$  be such that  $Gk$  is not the equalizer of  $Gf$  and  $Gg$ . As  $kf = kg$ , we have  $Gf Gk = Gg Gk$  and  $Im Gk \subset \{t \in GY, Gf(t) = Gg(t)\}$ . As clearly these sets are not equal, there exists  $t \in GY - Im Gk$  with  $Gf(t) = Gg(t) = u$ . Clearly  $X \neq \emptyset$ , so  $FX \neq \emptyset$ . Let  $\omega^X: FX \rightarrow GX$  be the constant onto  $\mu$ ,  $\omega^Y: FY \rightarrow GY$  the constant onto  $t$ . Clearly  $f, g: \langle X, \omega^X \rangle \longrightarrow \langle Y, \omega^Y \rangle$ . Let  $l: \langle Y, \omega^Y \rangle \longrightarrow \langle Z, \omega^Z \rangle$  be their coequalizer in  $A(F, G, 1)$ . Clearly  $Z \neq \emptyset$  and so  $FZ \neq \emptyset$ , let

$x \in F\mathbb{Z}$ ,  $t = \omega^Y Fl(x) = Gl(\omega^Z(x))$  and so  
 $t \in Im Gl$ . But as  $lf = lg$ ,  $l$  is coarser than  
 $h$ ,  $l = \tau h$  and so  $Gl = Gh \circ G\tau$  from which it im-  
 mediately follows that  $Im Gl \subset Im Gh$  and that  
 is a contradiction.

3)  $\implies$  2). The case that  $F = C_{O, M}^*$  is trivial. If  $G$   
 turns coequalizers into equalizers, we proceed analogously  
 as in 2.1.

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