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NON-CONSTANT CONTINUOUS MAPPINGS OF METRIC OR COMPACT HAUSDORFF SPACES

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The aim of the present note is to state and to prove the following theorems:

Theorem 1. There exists a class $\mathbb{M}$ of connected metric spaces such that all the spaces from $\mathbb{M}$ together with all their non-constant continuous mappings form a category that is isomorphic to the category $\mathcal{G}$ of all graphs. Every continuous mapping between the elements of $\mathbb{M}$ is a contraction x).

Theorem 2. Let there be no measurable cardinal. Then there exists a class $\mathbb{K}$ of compact Hausdorff spaces such that all the spaces from $\mathbb{K}$ with all their non-constant continuous mappings form a category isomorphic to the category $\mathcal{G}$ of all graphs.

Theorem 3. There exists a class $\mathbb{L}$ of metric continua such that all the spaces from $\mathbb{L}$ and all their non-constant continuous mappings form a category isomorphic to the category $\mathcal{G}_f$ of all finite graphs. Every continuous

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x) A mapping $f:(M,\varphi) \to (M',\varphi')$ is said to be a contraction iff $\varphi'(f(x), f(y)) \leq \varphi(x, y)$ always.

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mapping between the elements of $\mathbb{L}$ is a contraction.

Corollaries. Denote by $\mathcal{C}at\ \mathbb{M}$ (or $\mathcal{C}at\ \mathbb{K}$ or $\mathcal{C}at\ \mathbb{L}$) the category of all spaces of $\mathbb{M}$ (or $\mathbb{K}$ or $\mathbb{L}$, respectively) and all their non-constant continuous mappings.

a) Since every algebraic category can be fully embedded in $\mathcal{G}$ (see [6]), it can be fully embedded in $\mathcal{C}at\ \mathbb{M}$.

b) Every small category can be fully embedded in $\mathcal{G}$ (see [8]), consequently in $\mathcal{C}at\ \mathbb{M}$. Particularly, every monoid can be represented as a monoid of all non-constant continuous mappings of a metric space into itself, which strengthens a result from [4].

c) If there is no proper class of measurable cardinals, then every concrete category can be fully embedded in $\mathcal{G}$ (see [5]), consequently in $\mathcal{C}at\ \mathbb{M}$. Particularly, a large discrete category can be fully embedded in $\mathcal{G}$ (proof see in [9]), consequently there exists a proper class of metric spaces such that every continuous mapping between two of them is either an identical mapping of a space onto itself or constant.

d) If there is no measurable cardinal then a) b) c) are true, replacing $\mathcal{C}at\ \mathbb{M}$ by $\mathcal{C}at\ \mathbb{K}$ and "metric space" by "compact Hausdorff space".

e) Every finite category can be fully embedded in $\mathcal{G}_f$ (proved implicitly in [8]), consequently in $\mathcal{C}at\ \mathbb{L}$. Especially, every finite monoid can be represented as a monoid of all non-constant continuous mappings of a metric continuum into itself.

f) Since every continuous mapping between the elements of $\mathbb{M}$ (or $\mathbb{L}$) is a contraction, every monoid (or finite monoid) can be represented as a monoid of all non-constant proximally continuous or uniformly continuous or Lipschitz mappings or contractions of a metric space (or metric continuum, respectively) into itself.

Proof of Theorem 1. I. We recall that $\mathcal{G}$ is the category, the objects of which are all graphs $G = (X, R)$ (i.e. X is a non-empty set, $R \subset X \times X$) and morphisms are all compatible mappings (i.e. if $G = (X, R)$, $G' = (X', R')$ are graphs, $f: G \longrightarrow G'$ is a morphism of $\mathcal{G}$ iff $f: X \longrightarrow X'$ is a mapping with $(f \times f)(R) \subset R'$). The category $\mathcal{G}$ is isomorphic to a full sub-category of the category $\mathcal{G}_c$ of all connected graphs without loops x) and all their compatible mappings (see [7]).
So we can prove Theorem 1 replacing $\mathcal{G}_c$ instead of $\mathcal{G}$ in it.

II. Lemma 1. Let a continuum H be a subspace of a Hausdorff space Q, $a, b \in H$, $a \neq b$. Let $M = H - \{a, b\}$ be an open subset of Q. Let Z be a continuum, $f: Z \longrightarrow Q$ be a continuous mapping. Then there exists either a component C of the set $f^{-1}(H)$ such that $a, b \in f(C)$ or a continuous mapping $\tilde{f}: Z \longrightarrow Q$

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x) We recall that a graph $G = (X, R)$ is said to be connected if for every $a, b \in X$ (not necessarily different) there exists $x_0, \dots, x_n$ such that $a = x_0$, $b = x_n$ and either $\langle x_{i-1}, x_i \rangle \in R$ or $\langle x_i, x_{i-1} \rangle \in R$, $i = 1, \dots, n$. Every pair $\langle x, x \rangle \in R$ is said to be a loop of G.

such that $\tilde{f}(x) = f(x)$ whenever $f(x) \in Q - M$, $\tilde{f}(x) \in \{a, b\}$ whenever $f(x) \in M$.

<u>Proof</u>. If either $a \notin f(Z)$ or $b \notin f(Z)$, then the lemma is trivial. Let $a, b \in f(Z)$. Let there exist no component C of $f^{-1}(H)$ with $a, b \in f(C)$. Put $A = f^{-1}(a)$, $B = f^{-1}(b)$.

1) We show that every component L of $f^{-1}(H)$ intersects $A \cup B$. Let L be a component of $f^{-1}(H)$ with $L \cap (A \cup B) = \emptyset$. Then there exists a closed-open subset G of $f^{-1}(H)$ such that $L \subset G \subset f^{-1}(H) - (A \cup B)$. Then G is closed in Z and, since G is also an open subset of an open $f^{-1}(M)$, G is open in Z. But Z is a continuum.

2) Denote by $\mathcal{L}_A$ (or $\mathcal{L}_B$) the system of all components of $f^{-1}(H)$ that intersect A (or B, respectively). Put $P_A = \bigcup \mathcal{L}_A$, $P_B = \bigcup \mathcal{L}_B$. 1) implies $f^{-1}(H) = P_A \cup P_B$ and $P_A \cap P_B = \emptyset$. We show that both P_A and P_B are open in $f^{-1}(H)$. If $x \in P_A$, then $x \in L$ for some component $L \in \mathcal{L}_A$. Then there exists a closed-open subset G of $f^{-1}(H)$ such that $L \subset G \subset f^{-1}(H) - B$. Then necessarily $G \subset P_A$, thus P_A is open.

3) Now define

$$\tilde{f}(x) = f(x \quad \text{whenever } f(x) \in Q - M,$$
$$\tilde{f}(x) = a \quad \text{whenever } x \in P_A,$$
$$\tilde{f}(x) = b \quad \text{whenever } x \in P_B.$$

One can see easily that $\tilde{f}$ is a continuous mapping, satisfying the required conditions.

III. Conventions. a) If $\mathcal{M}$ is a metric space, $|\mathcal{M}|$ denotes its underlying set.

b) Let $\mathcal{M}$ be a bounded metric space with a metric α and a diameter d. Let R be a set, ℓ be a real number, $\ell \geq d$. Then by $\bigvee_{r \in R}^{\ell} (\mathcal{M} \times \{r\})$ we denote the metric space with the underlying set $\bigcup_{r \in R} (|\mathcal{M}| \times \{r\})$ and the metric, say σ, defined as follows:

$\sigma(\langle x, r\rangle, \langle y, r\rangle) = \alpha(x, y)$, $\sigma(\langle x, r\rangle, \langle y, r'\rangle) = \ell$ whenever $r \neq r'$.

c) Let $\mathcal{M} = (|\mathcal{M}|, \alpha)$, $\mathcal{M}' = (|\mathcal{M}'|, \alpha')$ be metric spaces, $\varphi : |\mathcal{M}| \longrightarrow |\mathcal{M}'|$ be a mapping onto $|\mathcal{M}'|$. We say that $\mathcal{M}'$ is a metric factor space of $\mathcal{M}$ given by φ whenever for every $x, y \in |\mathcal{M}'|$ $\alpha'(x, y) = $ $= \inf \sum_{i=0}^{n} \alpha(a_i, b_i)$, where the infimum is taken over all chains $(a_0, b_0, \ldots, a_n, b_n)$ such that $\varphi(a_0) = x$, $\varphi(b_n) = $ $= y$ and $\varphi(b_{i-1}) = \varphi(a_i)$, $i = 1, \ldots, n$. In fact, $\mathcal{M}'$ is a factor-object of $\mathcal{M}$ in the category of metric spaces and contractions.

d) In [1] a space $\mathcal{M}_1$ with the following properties is constructed:

$\mathcal{M}_1$ is a metric continuum;

if Z is a sub-continuum of $\mathcal{M}_1$, $f : Z \longrightarrow \mathcal{M}_1$ is a continuous mapping, then either f is constant or $f(x) = $ $= x$ for all $x \in Z$.

The symbol $\mathcal{M}_1$ is kept for this space, φ for its metric, d for its diameter in the sequel. The subspaces of $\mathcal{M}_1$ are always considered as metric spaces with a restriction of φ.

e) Let H, K_1, K_2 be three pairwise disjoint subcontinua of M_1 that will be fixed in the sequel. Then the following is true for the subspace $H \cup K_1 \cup K_2$ of M_1:

($*$) If $Z \subset H \cup K_1 \cup K_2$ is a continuum, $f: Z \longrightarrow$ $\longrightarrow H \cup K_1 \cup K_2$ is a continuous mapping, then either f is constant or $f(x) = x$ for all $x \in Z$.

IV. To prove Theorem 1, we shall construct, for every connected graph G without loops, a metric space P_G ($\mathbb{M}$, then, will be the class of all these P_G). First, using an idea from [3] a space Q_G (a subspace of the P_G described later) is constructed replacing the arrows of G by issues of H. More precisely:

Choose $a, b \in H$, $a \neq b$. Let a connected graph without loops $G = (X, R)$ be given; denote by π_1 or π_2 the first or the second projection.

The metric space Q_G is defined as follows: Let

$$\varphi: \bigcup_{r \in R} (|H| \times \{r\}) \longrightarrow |Q_G|$$

be the factor mapping defined by the following equalities: $\varphi(\langle b, r \rangle) = \varphi(\langle a, r' \rangle)$ whenever $r, r' \in R$, $\pi_2(r) = \pi_1(r')$. Let Q_G be a metric factor space of $\bigvee^{d}_{r \in R}(H \times \{r\})$ given by φ. For every $r \in R$, $x \in H$ put $x_r = \varphi(\langle x, r \rangle)$. The set $T = \{a_r ; r \in R\} \cup \{b_r ; r \in R\}$ is a closed discrete subset of Q_G.

<u>Lemma 2</u>. Let either $Z = H$ or $Z = K_1$ or $Z = K_2$, $f: Z \longrightarrow Q_G$ be a continuous mapping. Then either f is constant or $Z = H$ and there exists $r \in R$ such that $f(x) = x_r$ for every $x \in Z$.

Proof. Put $H_r = \varphi(H \times \{r\})$. If $t \in T$ put $A_t = \bigcup_{t \in H_r} H_r$, $St_t = (A_t - T) \cup \{t\}$, $D_t = A_t \cap T$.

Put $S = T \cap f(Z)$.

1) If $S = \emptyset$, then, since $f(Z)$ is connected and $(*)$ holds, f is constant.

2) Let $\mathit{card}\ S = 1$, say $S = \{s\}$. Since $f(Z)$ is connected, then $f(Z) \subset St_s$, i.e. $f = i \circ f'$ where $i : St_s \longrightarrow Q_G$ is the inclusion. We prove that f is a constant to s. If there exists $y \in St_s - \{s\}$, $y \in f(Z)$, define the mapping $g : St_s \longrightarrow St_s$ such that

$g(x) = x$ whenever $x \in H_{r_0} - T$ where r_0 is the element of R with $y \in H_{r_0}$,

$g(x) = s$ otherwise.

g is continuous and $(*)$ implies that $g \circ f'$ is constant, which is a contradiction.

3) Let $\mathit{card}\ S > 1$. One can see easily that the mapping $g : Z \longrightarrow Q_G$ such that

$g(x) = f(x)$ whenever $f(x) \in H_r$ with a_r, $b_r \in f(Z)$,

$g(x) = a_r$ whenever $f(x) \in H_r$, $b_r \notin f(Z)$,

$g(x) = b_r$ whenever $f(x) \in H_r$, $a_r \notin f(Z)$

is continuous. Since Z is compact, the set $S = f(Z) \cap \cap T = g(Z) \cap T$ is finite. Let $L = \{\ell_1, \dots, \ell_m\}$ be the set of all triples $\ell_i = \langle s_i, s_i', H_{r_i} \rangle$ such that $s_i, s_i' \in S$, $s_i \neq s_i'$, $r_i \in R$, s_i, $s_i' \in H_{r_i}$, and there exists no component C of the set $g^{-1}(H_{r_i})$ with $s_i, s_i' \in f(C)$. Now we use Lemma 1 n-times, we put $q_0 = q$, $q_{i+1} = \tilde{q}_i$. The continuous mapping $q_m : Z \longrightarrow Q_G$ has the following property:

If for some $r \in R$ the set $q_m(Z) \cap H_r$ is non-empty, then either

a) $q_m(Z) \cap H_r \subset \{a_r, b_r\}$ or

b) there exists a component C of $q_m^{-1}(H_r)$ such that $a_r, b_r \in q_m(C)$.

Since $q_m(Z)$ is connected, then necessarily there exists $r_0 \in R$ such that b) holds for it. Then $(*)$ implies $Z = H$ and $q_m(x) = x_{r_0}$ for all $x \in C$. Particularly, $q_m(a) = a_{r_0}$, $q_m(b) = b_{r_0}$, i.e. $a, b \in C$. Consequently, there exists exactly one such r_0. Since $q_m(Z)$ is connected, $q_m(Z) \subset H_{r_0}$. Then $(*)$ implies $q_m(x) = x_{r_0}$ for all $x \in Z = H$. Then, clearly, $q_m = q_{m-1} = \ldots = q_0 = q = f$.

V. Let H, K_1, K_2, a, b have the same meaning as in IV. Moreover, choose $c_1, c_2 \in H$ such that $\operatorname{card}\{a, b, c_1, c_2\} = 4$ and choose $p_i, d_i \in K_i$, $i = 1, 2$, $p_i \neq d_i$. The metric space P_G is defined as follows: Let

$$\psi : \bigcup_{r \in R} (|H \cup K_1 \cup K_2| \times \{r\}) \longrightarrow |P_G|$$

be the factor mapping defined by the following equalities:

$\psi(\langle b, r\rangle) = \psi(\langle a, r'\rangle)$ whenever $r, r' \in R$, $\pi_2(r) = \pi_1(r')$;

$\psi(\langle d_i, r\rangle) = \psi(\langle c_i, r\rangle)$ whenever $r \in R$, $i = 1, 2$;

$\psi(\langle p_1, r\rangle) = \psi(\langle p_2, r'\rangle)$ whenever $r, r' \in R$.

The space P_G is the metric factor space of $\bigvee^{d}_{\kappa \in R} ((H \cup K_1 \cup K_2) \times \{\kappa\})$ given by ψ. The space Q_G is a subspace of P_G and ψ is an extension of φ. Put $H_\kappa = \psi(H \times \{\kappa\})$, $K_{i\kappa} = \psi(K_i \times \{\kappa\})$, $y_\kappa = \psi(\langle y, \kappa \rangle)$. The point $p_{1\kappa} = p_{2\kappa}$ will be also denoted by p_G. Put $T_G = \{a_\kappa ; \kappa \in R\} \cup \{b_\kappa ; \kappa \in R\}$, $D_i = \{d_{i\kappa} ; \kappa \in R\}$, $i = 1, 2$. Clearly, $T_G \cup D_1 \cup D_2$ is a closed discrete subset of P_G and there is a bijection

$$\lambda_G : X \longrightarrow T_G$$

onto T_G such that for every $x \in X$ either $\lambda_G(x) = a_\kappa$ where $\pi_1(\kappa) = x$, or $\lambda_G(x) = b_\kappa$ where $\pi_2(\kappa) = x$.

<u>Lemma 3</u>. Let either $Z = H$ or $Z = K_1$ or $Z = K_2$. Let $f : Z \longrightarrow P_G$ be a continuous mapping. Then either f is constant or there exists $\kappa \in R$ such that $f(x) = x_\kappa$ for all $x \in Z$.

<u>Proof</u>. 1) Let $p_G \notin f(Z)$. Then use the retraction $q : P_G - \{p_G\} \longrightarrow Q_G$ with $q(K_{i\kappa} - \{p_G\}) = \{d_{i\kappa}\}$, Lemma 2 and ($*$).

2) Let $p_G \in f(Z)$. If $f(Z) \cap (D_1 \cup D_2) = \emptyset$, then f is constant. (It may be proved analogously to 2) in the proof of Lemma 2.) Let $S = f(Z) \cap (D_1 \cup D_2) \neq \emptyset$. Define $g : Z \longrightarrow P_G$ as follows: $g(x) = f(x)$ whenever $f(x) \in Q_G$ or $(f(x) \in K_{i\kappa})$ & $(d_{i\kappa} \in f(Z))$, $g(x) = p_G$ otherwise.
Then g is continuous, $g(Z) \cap (D_1 \cup D_2)$ is finite. Let $L_i = \{t^i_1, \dots, t^i_{m_i}\}$ be the set of all

points of $g(Z) \cap D_i$ such that for no component C of $g^{-1}(K_{ir})$ is $p_G, d_{ir} \in f(C)$ $(i = 1, 2)$. We use Lemma 1 $(m_1 + m_2)$-times and we obtain a continuous mapping $h : Z \longrightarrow P_G$ with the following property: if $r \in R$, $i \in \{1, 2\}$, then

a) either $h(Z) \cap K_{ir} \subset \{p, d_{ir}\}$ or

b) there exists a component C of the set $h^{-1}(K_{ir})$ such that $p_G, d_{ir} \in h(C)$. One can see easily (analogously to the proof of Lemma 2) that the case b) is true precisely for one couple $\langle r_0, i_0 \rangle \in R \times \{1, 2\}$. Define a mapping $\ell : Z \longrightarrow K_{i_0 r_0}$ such that $\ell(x) = h(x)$ whenever $h(x) \in K_{i_0 r_0}$, $\ell(x) = d_{i_0 r_0}$ otherwise. Since ℓ is continuous non-constant, then necessarily $Z = K_{i_0}$ and $\ell(x) = x_{r_0}$ for all $x \in Z$. But then $\ell = h = g = f$.

VI. Let $G = (X, R)$, $G' = (X', R')$ be connected graphs without loops, $f : G \longrightarrow G'$ be a compatible mapping. Define a mapping $\bar{f} : P_G \longrightarrow P_G$ as follows: if $r = \langle r_1, r_2 \rangle \in R$, $x \in H \cup K_1 \cup K_2$, put $\bar{f}(x_r) = x_{r'}$ where $r' = \langle f(r_1), f(r_2) \rangle \in R'$. It is easy to see that every $\bar{f}$ is a non-constant contraction. Conversely, let $g : P_G \longrightarrow P_{G'}$ be a continuous mapping. We want to prove that either g is constant or $g = \bar{f}$ for some compatible mapping $f : G \longrightarrow G'$.

1) First we prove: If there exists $r \in R$ such that the restriction g/H_r or g/K_{1r} or g/K_{2r} is constant, then g is constant. But it follows easily from

Lemma 3 and the fact that G is connected.(To prove it denote by s the value of φ/H_r (or φ/K_{1r} or φ/K_{2r} respectively) and discuss the cases $s = p$, $s \in Q_G$, $s \in P_G - \{Q_G \cup p\}$.)

2) If g is not constant, then for every $r \in R$ there exists $r' \in R'$ such that $g(x) = x_{r'}$ for all $x \in H$. Then necessarily $g(T_G) \subset T_{G'}$. If we put $f = \lambda_{G'}^{-1} \circ g \circ \lambda_G$ then $f : G \longrightarrow G'$ is a compatible mapping and $g = \bar{f}$.

VII. Now it is evident that the class $\mathbb{M}$ of all the spaces P_G , where G runs over all connected graphs without loops, has the required properties.

Proof of Theorem 3 is, in fact, the same as the proof of Theorem 1. It is only necessary to notice that the category $\mathcal{G}_f$ of all finite graphs is isomorphic to a full subcategory of the category $\mathcal{G}_{fc}$ of all finite connected graphs without loops (proved implicitly in [7]). If G is a finite connected graph without loops, then clearly the space P_G is a metric continuum.

Proof of Theorem 2.

I. Lemma 4. Let M be a realcompact metric space, $x \in \beta M - M$. Let $x_n \in \beta M$, $x = \lim_{n \to \infty} x_n$. Then there exists a natural number n_0 , such that $x_n = x$ for all $n \geq n_0$.

Proof. It follows immediately from Theorem 9.11 in [2].

II. Lemma 5. Let M, M' be metric spaces, M connected, M' realcompact. Let $\varphi : \beta M \longrightarrow \beta M'$ be a continuous mapping. Then either φ is constant or $\varphi(M) \subset M'$.

Proof. Let $\varphi(x) \in \beta M' - M'$ for some $x \in M$. Put $A = M \cap \varphi^{-1}(\varphi(x))$. A is a closed subset of M and Lemma 4 implies that A is open. So $A = M$, φ is constant.

III. If there is no measurable cardinal, then every metric space is realcompact. Then it is easy to see that the class $K = \{\beta M ; M \in \mathbb{M}\}$ has all the required properties.

References

[1] H. COOK: Continua which admit only the identity mapping onto non-degenerate subcontinua, Fund.Math. 60(1966),241-249.

[2] L. GILLMAN, M. JERISON: Rings of continuous functions, Van Nostrand's University series in higher mathematics.

[3] J. de GROOT: Groups represented by homeomorphism groups I, Math.Annalen 138(1959),80-102.

[4] Z. HEDRLÍN: Non-constant continuous transformations form any semigroup with unity, Nieuw Archief voor Wiskunde (3),XIV,230-236(1966).

[5] Z. HEDRLÍN: Extensions of structures and full embeddings of categories, Actes,Congrès intern.math., 1970,Tome 1,319-322.

[6] Z. HEDRLÍN, A. PULTR: On full embeddings of categories of algebras, Illinois J.of Math.10(1966),392-405.

[7] Z. HEDRLÍN, J. LAMBEK: How comprehensive is the category of semigroups,J.of Algebra 11(1969), 195-212.

[8] Z. HEDRLÍN, A. PULTR: O predstavlenii malych kategorij, DAN SSSR 160,284-286(1965).

[9] Z. HEDRLÍN, P. VOPĚNKA: An undecidable theorem concerning full embeddings into categories of algebras, Comment.Math.Univ.Carolinae 7(1966), 401-409.

[10] A. PULTR: Concerning universal categories, Comment. Math.Univ.Carolinae 5(1964),227-239.

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