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PLANAR PERMUTATION GRAPHS OF PATHS

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The present note gives the solution of the first of three problems stated in Chartrand and Frechen [2]; formerly, this problem appears in an implicite form in Chartrand and Harary [1]. (This problem has a certain relation to the question concerning mathematical linguistics discussed in [3].)

Let $n \geq 1$. Consider a path A_n with the set of vertices $R = \{r_1, \dots, r_{n+1}\}$ and the set of edges $E_R = \{r_1 r_2, \dots, r_n r_{n+1}\}$. By B_n we shall denote a disjoint copy of the path A_n such that B_n has the set of vertices $S = \{s_1, \dots, s_{n+1}\}$ and the set of edges $E_S = \{s_1 s_2, \dots, s_n s_{n+1}\}$. Let α be a permutation on the set $\{1, \dots, n+1\}$. By $P_\alpha(A_n)$ we denote the graph with the set of vertices $R \cup S$ and the set of edges $E_R \cup E_S \cup \{r_1 s_{\alpha(1)}, \dots, r_{n+1} s_{\alpha(n+1)}\}$. The graph $P_\alpha(A_n)$ is a special case of permutation graphs which were studied in [1] and [2].

Integers will be denoted by e, f, g, h, i, j and k. We shall write $med(f, g, h)$ if either $f < g < h$

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or $h < g < f$.

Theorem. A necessary and sufficient condition for $P_\alpha(A_n)$ to be planar, is that for any i, j, k such that $1 < i < j < k < n+1$, at most one of the following two statements hold:

(1) $$\mathrm{med}\,(\alpha(i), \alpha(j), \alpha(i-1)),$$

(2) $$\mathrm{med}\,(\alpha(k), \alpha(j), \alpha(k+1)).$$

Proof. Necessity: Assume that $P_\alpha(A_n)$ is planar and that there exist i, j, k such that $1 < i < j < k \leq n$ and both (1) and (2) hold. Let e, f, g, h be such that $\{e, f, g, h\} = \{i, i-1, k, k+1\}$ and $\alpha(e) < \alpha(f) < \alpha(j) < \alpha(g) < \alpha(h)$. By G we denote the subgraph of $P_\alpha(A_n)$ consisting of the path between r_{i-1} and r_{k+1} in A_n, the path between $s_{\alpha(e)}$ and $s_{\alpha(h)}$ in B_n, and the edges $r_e s_{\alpha(e)}$, $r_f s_{\alpha(f)}$, $r_j s_{\alpha(j)}$, $r_g s_{\alpha(g)}$, $r_h s_{\alpha(h)}$. Obviously, G is homeomorphic to the complete bipartite graph $K_{3,3}$; the vertices r_j, $s_{\alpha(f)}$, $s_{\alpha(g)}$ and r_i, r_k, $s_{\alpha(j)}$ represent its two levels. Thus $P_\alpha(A_n)$ is not planar, which is a contradiction.

Sufficiency: Consider a cartesian plane. For every j, $1 \leq j \leq n+1$, we define the points $v_j = (j, \alpha(j))$, $w_j = (0, j)$ and $z_j = (n+2, j)$. We shall say that a point v_j is of the first or the second kind if there exist k, $1 \leq k \leq n$, such that the intervals $v_j z_{\alpha(j)}$ and $v_k v_{k+1}$ or the intervals $v_j w_{\alpha(j)}$ and $v_k v_{k+1}$, respectively, cross. It is readily seen that no point v_j is simultaneously of the first and of the second kind. We shall say that a point v_j is of the third kind if it is neither of the first nor of the second kind. The graph $P_\alpha(A_m)$ can be embedded in the plane as follows: every vertex r_e is drawn as the point v_e ; every vertex s_f is drawn as the point w_f ; every edge $r_g r_{g+1}$ as the interval $v_g v_{g+1}$; every edge $s_h s_{h+1}$ as the interval $w_h w_{h+1}$; every edge $r_i s_{\alpha(i)}$ as the interval $v_i w_{\alpha(i)}$, when v_i is of the first or the third kind and as a suitable arc passing through the point $x_{\alpha(i)}$, when v_i is of the second kind. Obviously, there are arcs C_j connecting w_j with x_j such that no two of them intersect and that C_j meets the oblong $\langle 0, \dots, m+2 \rangle \times \langle 1, \dots, m+1 \rangle$ only in w_j and x_j. Thus, it suffices to extend the intervals $v_i x_{\alpha(i)}$ for v_i of the second kind by $C_{\alpha(i)}$.

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References

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