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ON EXISTENCE OF THE WEAK SOLUTION FOR NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF ELLIPTIC TYPE, II.

J. KAČUR, Bratislava

This paper is a direct continuation of my paper [1] concerning the existence of a weak solution of boundary value problems for non-linear elliptic equations of the form

$$\sum_{i \in M} (-1)^{|i|} D^i a_i(x, D^j u) = f$$

in Orlicz-Sobolev spaces. Therefore, to follows this paper, we have to make use of [1]. The used notation is in accordance with [1] and the numbering of paragraphs, theorems and relations is being continued as well. The used fundamental notions are defined in [1]. The main aim of our paper is to prove the fact that it is sufficient to assume the algebraic condition (2.16), i.e.,

$$\sum_{i \in M} \xi_i a_i(x, \xi_j) \geq c_1 \sum_{i \in M} \xi_i q_i(\xi_i) - c_2$$

to guarantee the coercivity (2.7), i.e.,

$$\lim_{\|u\|_{W^k_G} \to \infty} \|u\|^{-1}_{W^k_G} \cdot \int_\Omega \sum_{i \in M} D^i u \, a_i(x, D^j(u_0 + u))\, dx = \infty ,$$

where $u_0 \in W^k_G$.

In the paper [1] we proved (2.7) assuming (2.16) and the rather limited assumption (1.9) which includes the following

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condition:

For all $i \in M$ there exist $r_i > 1$, $s_i > 1$ with $0 < r_i - s_i < 1$ so that

$$c_{1i} |u|^{s_i} \leq u g_i(u) \leq c_{2i} |u|^{r_i}$$

for all $|u| \geq u_i > 0$, where c_{1i}, c_{2i}, u_i are the suitable constants.

In many cases, the condition (2.16) can yet be weakened. In this connection a theorem about the equivalence of norms is proved (Theorem 10), which itself is also interesting. As a consequence of these results we obtain existence theorems for the weak solution with hypotheses that can be easily verified in concrete problems.

In the next remark we call the attention to the fact that the class $\mathcal{M}_3$ by means of which the non-linear members are described is essentially larger than the set of polynomials $|u|^p$.

Remark. If $g(u) \in \mathcal{M}_3$, then Assertion 1, § 1 guarantees the existence of $p > 1$, $q > 1$ such that (1.1), i.e.,

$$c_1 |u|^p \leq u g(u) \leq c_2 |u|^q \quad \text{for all } |u| \geq c$$

holds, where c_1, c_2, c are the suitable constants. On the contrary, for all p, q with $q > p > 1$, there exists $g_{p,q}(u) \in \mathcal{M}_3$ such that (1.1) holds, while the relation (1.1) does not take place for any p', q' with $p < p' < q' < q$.

We shall denote positive constants by c with or without subscripts and the dependence of c on the parameter ε will be denoted by $c(\varepsilon)$.

§ 5.

Let $u_0(x)$ be a function in $W^{\kappa}_{\vec{G}}(\Omega)$. ($u_0(x)$ represents the stable boundary values - see p. 153.)

Our main result is

Theorem 7. If the conditions (2.2) and (2.16) are fulfilled, then (2.7) holds.

Proof. From (2.16) we obtain

$$
\begin{aligned}
&\int_\Omega \sum_{i\in M} D^i u\, a_i(x, D^j(u_0+u))\,dx = \\
&\qquad = \int_\Omega \sum_{i\in M} D^i(u_0+u)\, a_i(x, D^j(u_0+u))\,dx - \\
&- \int_\Omega \sum_{i\in M} D^i u_0\, a_i(x, D^j(u_0+u))\,dx \geq \\
&\qquad \geq c_1 \sum_{i\in M} \int_\Omega D^i(u_0+u)\, q_i(D^i(u_0+u))\,dx - \\
&- \int_\Omega \sum_{i\in M} D^i u_0\, a_i(x, D^j(u_0+u))\,dx - \\
&\qquad - c_2 \geq c_1 \sum_{i\in M} \int_\Omega G_i(D^i(u_0+u))\,dx - \\
&- \int_\Omega \sum_{i\in M} D^i u_0\, a_i(x, D^j(u_0+u))\,dx - c_3 .
\end{aligned}
\tag{5.1}
$$

In the last inequality, we have used the evident estimation

$$
-c_i + u\, q_i(u) \leq G_i(u) \leq u\, q_i(u) + c_i'
$$

for all u, since $p.p.\ G_i(u) = u\, q_i(u)$ - see § 1. Now, with the help of the Young's inequality and using the convexity of N-functions $P_i(v)$ we estimate

$$
\begin{aligned}
&\sum_{i\in M} \int_\Omega \frac{D^i u_0}{\varepsilon}\, \varepsilon\, a_i(x, D^j(u_0+u))\,dx \leq \\
&\qquad \leq \sum_{i\in M} \int_\Omega G_i\left(\frac{D^i u_0}{\varepsilon}\right) dx + \\
&\qquad + \sum_{i\in M} \int_\Omega P_i(\varepsilon\, a_i(x, D^j(u_0+u)))\,dx \leq
\end{aligned}
\tag{5.2}
$$

$$\leq c_1(u_0, \varepsilon) + \varepsilon \sum_{i \in M} \int_\Omega P_i(a_i(x, D^j(u_0 + u)))\, dx$$

where $\varepsilon \in (0, 1)$. Again by the convexity, together with the Δ_2-condition and (2.2), we successively obtain

$$P_i(a_i(x, \xi_j)) \leq$$

$$(5.3) \quad \leq \frac{1}{\varkappa} \sum_{j \in M} P_i(\varkappa . c \min(|q_i(\xi_j)|, |q_j(\xi_j)|) + \varkappa . c) \leq$$

$$\leq c_4 \sum_{j \in M} P_i(\min(|q_i(\xi_j)|, |q_j(\xi_j)|)) + c_5 ,$$

where $\varkappa = \operatorname{card} M + 1$.

In § 2 (proof of Lemma 1) the inequality

$$\min(|q_i(u)|, |q_j(u)|) \leq 2\, q_i(G_i^{-1}(G_j(u)))$$

is proved for each $|u| \geq c_6$, $i, j \in M$. ($G_i^{-1}(u)$ is the inverse function to $G_i(u)$ for $u \geq 0$.) From this inequality and owing to (1.4), i.e.,

$$P_i(q_i(u)) \leq G_i(u) \quad \text{for each } |u| \geq c_7,\ i \in M,$$

we deduce, using the Δ_2-condition

$$(5.4) \quad P_i(\min(|q_i(\xi_j)|, |q_j(\xi_j)|)) \leq c(2) G_j(\xi_j) + c_8 .$$

From the inequalities (5.3) and (5.4) we conclude

$$(5.5) \quad \sum_{i \in M} \int_\Omega P_i(a_i(x, D^j(u_0 + u)))\, dx \leq$$

$$\leq c_9 \sum_{i \in M} \int_\Omega G_i(D^i(u_0 + u))\, dx + c_{10} .$$

In the relation (5.2), we choose $\varepsilon \in (0, 1)$ such that $c_1 - \varepsilon c_9 = c_{11} > 0$. Then, from (5.1), (5.2) and (5.5) we have

$$\sum_{i \in M} \int_\Omega D^i u \, a_i(x, D^j(u_0 + u))\, dx \geq$$

$$(5.6) \qquad \geq c_{11} \sum_{i \in M} \int_\Omega G_i(D^i(u_0 + u))\, dx - c_2(u_0, e) \, .$$

Finally, it follows from Theorem 1, § 1

$$\lim_{\|u\|_{\overset{\circ}{W}{}^{m}_{\vec{G}}} \to \infty} \|u_0 + u\|^{-1}_{W^{m}_{\vec{G}}} \cdot \int_\Omega \sum_{i \in M} G_i(D^i(u_0 + u))\, dx = \infty \, ,$$

if $(0, \dots, 0) \in M$. In the case $(0, \dots, 0) \notin M$, we consider $u \in \overset{\circ}{W}{}^{m}_{\vec{G}}(\Omega)$. Then, using the Young's inequality and applying Lemma 4, § 1 we estimate

$$\int_\Omega |u|\, dx \leq \int_\Omega G_i(|u|)\, dx + c_{12} \leq c_{13} \int_\Omega G_i(D^i u)\, dx + c_{14}$$

for some $i \in M$ from which it follows that the foregoing assertion is true and hence owing to (5.6) the proof of the theorem is complete.

In the following we establish some assertions in which the condition (2.16) will be weakened by means of assumptions of monotonicity and equivalence of norms. Now, let K, L, M, M_1 and M_2 from § 2 denote the sets of indices defined in § 2 (p.151 and p. 155). For the multiindices $i \equiv (i_1, \dots, i_N)$, $j \equiv (j_1, \dots, j_N)$ we denote $i \geq j$, iff $i_\ell \geq j_\ell$ for all $\ell = 1, 2, \dots, N$.

We shall weaken the condition (2.16) in the following way:

$$(5.7) \qquad \sum_{i \in M} \xi_i a_i(x, \xi_j) \geq c_1 \sum_{i \in M_1} \xi_i g_i(\xi_i) - c_2 \, .$$

In the case of non-Dirichlet problem we suppose that $(0, \dots, 0) \in M_1$

Moreover, we assume

For each $i \in M_2$ there exists $i' \in M_1$ such that

(5.8)

$$i' \geq i \quad \text{and} \quad G_i(u) \leq G_{i'}(u) \quad \text{for each } |u| \geq c \, .$$

Theorem 8. Let the conditions (2.2), (5.7) and (5.8) be fulfilled. Then, the relation (2.7) holds under the assumption $u \in \overset{\circ}{W}{}^{k}_{\vec{G}}(\Omega)$.

Proof. Similarly as in the proof of Theorem 7, we obtain

$$\sum_{i \in M} \int_\Omega D^i u \, a_i(x, D^j(u_0 + u))\, dx \geq$$

$$(5.9) \qquad \geq c_3 \sum_{i \in M_1} \int_\Omega G_i(D^i(u_0 + u))\, dx -$$

$$- \sum_{i \in M} \int_\Omega D^i u_0 \, a_i(x, D^j(u_0 + u))\, dx - c_4 \quad .$$

In § 1 (proof of Lemma 4) the estimation

$$\int_\Omega G(u(x))\, dx \leq c_5 \int_\Omega G\left(\frac{\partial u}{\partial x_i}\right) dx + c_6$$

is proved for $u \in \overset{\circ}{W}{}^{k}_{\vec{G}}$ and $i = 1, 2, \dots, N$, where $G(u)$ is the N-function satisfying the Δ_2-condition. By iteration of the last inequality and with the help of (5.8) we obtain for each $i \in M_2$

$$\int_\Omega G_i(D^i u)\, dx \leq \int_\Omega G_{i'}(D^i u)\, dx + c_7 \leq c_8 \int_\Omega G_{i'}(D^{i'} u)\, dx + c_9 \, .$$

Hence, due to the convexity and the Δ_2-condition, we have

$$\int_\Omega G_i(D^i(u_0 + u))\, dx \leq \frac{1}{2} \int_\Omega G_i(2 D^i u)\, dx +$$

$$+ \frac{1}{2} \int_\Omega G_i(2 D^i u_0)\, dx \leq c_{10} \int_\Omega G_i(D^i u)\, dx + c_{11} \leq$$

$$\leq c_{12} \int_\Omega G_{i'}(D^{i'}(u_0 + u))\, dx + c_{13} \quad .$$

In view of these estimations the relation (5.9) implies

$$\sum_{i\in M}\int_\Omega D^i u\, a_i(x, D^j(u_0+u))\,dx \geq c_{14}\sum_{i\in M}\int_\Omega G_i(D^i(u_0+u))\,dx -$$
$$-\sum_{i\in M}\int_\Omega D^i u_0\, a_i(x, D^j(u_0+u))\,dx - c_{15}\ .$$

From the last inequality the assertion of the theorem follows by the same argument as in the proof of Theorem 7.

In the following theorem we shall suppose that

$$(5.10)\quad \sum_{i\in M_1}\xi_i\, a_i(x, \xi_j) \geq c_1\sum_{i\in M_1}\xi_i\, g_i(\xi_i) - c_2\ .$$

In the case of the non-Dirichlet problem we suppose, in addition, that $(0,\dots,0)\in M_1$.

$$(5.11)\qquad \sum_{i\in M_2}(\xi_i - \tau_i)\,[a_i(x,\xi_j) - a_i(x,\tau_j)] \geq 0\ .$$

$$(5.12)\qquad \sum_{i\in M_2}\|D^i u\|_{G_i} \leq c\sum_{i\in M_1}\|D^i u\|_{G_i}$$

for $u\in W^m_{\vec G}(\Omega)$.

<u>Theorem 9</u>. Let the conditions (2.2),(5.10),(5.11) and (5.12) be satisfied. Further, let $a_i(x,\xi_j)$ for $i\in M_1$ be independent on ξ_j , $j\in M_2$. Then (2.7) holds.

<u>Proof</u>. From the condition (5.10) it follows

$$\sum_{i\in M}\int_\Omega D^i u\, a_i(x, D^j(u_0+u))\,dx \geq c_3\sum_{i\in M_1}\int_\Omega G_i(D^i(u_0+u))\,dx -$$
$$-\sum_{i\in M_1}\int_\Omega D^i u_0\, a_i(x, D^j(u_0+u))\,dx +$$
$$+\sum_{i\in M_2}\int_\Omega D^i u\, a_i(x, D^j(u_0+u))\,dx\ .$$

Similarly as in the proof of Theorem 7, by the estimation of the second member on the R.H.S. we obtain

$$\sum_{i\in M}\int_\Omega D^i u\, a_i(x, D^j(u_0+u))\,dx \geq$$
$$(5.13)\qquad \geq c_1(\varepsilon)\cdot\sum_{i\in M_1}\int_\Omega G_i(D^i(u_0+u))\,dx \; +$$
$$+\sum_{i\in M_2}\int_\Omega D^i u\, a_i(x, D^j(u_0+u))\,dx - c_2(\varepsilon)\;.$$

Using the Hölder's inequality we estimate

$$\sum_{i\in M_2}\int_\Omega D^i u\, a_i(x, D^j u_0)\,dx \leq \sum_{i\in M_2}\|D^i u\|_{G_i}\cdot\|a_i(x, D^j u_0)\|_{P_i} \leq$$
$$\leq c(u_0)\sum_{i\in M_2}\|D^i u\|_{G_i}$$

and hence with respect to (5.11), it follows from (5.13)

$$\sum_{i\in M}\int_\Omega D^i u\, a_i(x, D^j(u_0+u))\,dx \geq$$
$$(5.14)\qquad \geq c_1(\varepsilon)\sum_{i\in M_1}\int_\Omega G_i(D^i(u_0+u))\,dx \; -$$
$$- c(u_0)\sum_{i\in M_2}\|D^i u\|_{G_i} - c_2(\varepsilon)\;.$$

If ε is sufficiently small, then $c_1(\varepsilon) > 0$. From (5.12) we deduce

$$(5.15)\qquad \lim_{\|u\|_{W^{k}_{\vec G}}\to\infty}\|u_0+u\|^{-1}_{W^{k}_{\vec G}}\;\cdot$$
$$\cdot\int_\Omega\sum_{i\in M_1}G_i(D^i(u_0+u))\,dx = \infty\;,$$

if $(0,\dots,0)\in M_1$ - see Theorem 1, § 1. In case $(0,\dots,0)\notin M_1$ we consider $u\in \mathring{W}^{k}_{\vec G}$ (in the Dirichlet problem). Then, similarly as in the proof of Theorem 7 we estimate

$$\int_\Omega |u(x)|\,dx \leq c_3\int_\Omega G_i(D^i u)\,dx + c_4 \leq$$
$$\leq c_5\int_\Omega G_i(D^i(u_0+u))\,dx + c_6\;,$$

where $i \in M_1$. Due to this estimation, (5.15) is true even in the case $(0, \dots 0) \notin M_1$. Finally, the assertion of the theorem follows from (5.15) and (5.14).

<u>Remark</u>. If $u_0(x) \equiv 0$, then (2.7) follows from the conditions (2.2),(5.7), and (5.12). The assertion is obvious.

In the following we establish a theorem in which we study the connection between the compactness of the imbedding and the equivalence of norms of the space $W^{k}_{\vec{G}}(\Omega)$.

We shall suppose the condition (2.9). Theorems of imbedding and compactness of imbedding of the space $W^{k}_{\vec{G}}$ are studied in [3]. (There $W^{k}_{\vec{G}}$ is considered, where $G_i(u) \equiv G_j(u)$ for all i, j with $|i|, |j| \leq k$.)

<u>Theorem 10</u>. If (2.9) is satisfied, then

$\sum_{i \in M_1} \| D^i u \|_{G_i} + \| u \|_{L_1(\Omega)}$ is an equivalent norm in the space $W^{k}_{\vec{G}}(\Omega)$, i.e.,

$$c_1 \| u \|_{W^k_{\vec{G}}} \leq \sum_{i \in M_1} \| D^i u \|_{G_i} + \| u \|_{L_1(\Omega)} \leq c_2 \| u \|_{W^k_{\vec{G}}} .$$

<u>Proof</u>. It is sufficient to prove the first inequality. We prove it by contradiction. Thus, there exists a sequence $\{u_m\}$ from $W^{k}_{\vec{G}}$ such that

$$(5.16) \quad \frac{1}{m} \| u_m \|_{W^k_{\vec{G}}} \geq \sum_{i \in M_1} \| D^i u_m \|_{G_i} + \| u_m \|_{L_1(\Omega)} .$$

We can suppose that $\| u_m \|_{W^k_{\vec{G}}} = 1$. From the sequence $\{u_m\}$ we can select a weakly convergent subsequence which we denote again by $\{u_m\}$, $u_m \rightharpoonup u \in W^{k}_{\vec{G}}$.

The relation (5.16) implies $\| D^i u_m \|_{G_i} \longrightarrow 0$ with

$n \to \infty$, for all $i \in M_1$, and hence in view of (2.9) it follows $u_m \to u$ with $n \to \infty$ in the norm of the space $W^{k}_{\vec{g}}(\Omega)$.

Now, it follows from (5.16) that $\| u \|_{L_1(\Omega)} = 0$ and hence $\| u \|_{W^{k}_{\vec{g}}} = 0$. On the other hand,

$$\| u \|_{W^{k}_{\vec{g}}} = \lim_{n \to \infty} \| u_n \|_{W^{k}_{\vec{g}}} = 1$$

which yields a contradiction and the theorem is proved.

§ 6.

The definition of a weak solution of a boundary value problem is given by the relation (2.3) in § 2 (p. 153).

Now, we present a modification of Theorem 3, § 2, assuming the simplified hypotheses.

Theorem 11. Let (2.2) be satisfied. Let us consider the following conditions:

I. The conditions (2.16) and (2.8) are fulfilled.

II. The conditions (2.16),(2.10) and (2.9) are fulfilled.

III. The conditions (5.10),(5.11),(2.9) and (2.10) are fulfilled and $a_i(x, \xi_j)$ for $i \in M_1$ is independent on ξ_j , $j \in M_2$.

If one of the conditions I, II, III holds, then there exists a solution of the problem (2.3).

Theorem 12. Let (2.2) be satisfied. Let us consider the following conditions:

IV. The conditions (5.7),(5.8) and (2.8) are fulfilled.

V. The conditions (5.7),(5.8),(2.9) and (2.10) are fulfilled.

If one of the conditions IV, V is satisfied, then there exists a solution of the Dirichlet problem (2.3).

For the uniqueness of the solution of the problem (2.3) it suffices to assume (2.8a) in Theorem 11 and Theorem 12.

Proof of Theorem 11 and Theorem 12. The proof of these theorems is the same as that of Theorem 3, § 2. It is sufficient to show that the hypotheses of the theorem 3, § 2 are fulfilled. Due to the results from § 5, (2.7) holds in each of the cases I, II, IV and V. In the case III the condition (2.9) implies (5.12) and hence (2.7) holds. Finally, it is necessary to show that in the cases II, III and V it holds (2.11a), i.e.,

$$\lim_{\sum_{i\in L}|\xi_i|\to\infty} \Big(\sum_{i\in L}|\xi_i| + \sum_{i\in L}|g_i(\xi_i)|\Big)^{-1} \cdot \sum_{i\in M_1} \xi_i a_i(x,\xi_j) = \infty$$

uniformly for ξ_ℓ, $\ell \in M - L$ in a bounded set and $x \in \Omega$.
In the case III the condition (5.10) implies (2.11a).
In the cases II and V let us substitute the vectors $\xi' \equiv (\xi_\alpha\ \mathcal{T}_\beta)$ where $\alpha \in M_1$ and $\beta \in M_2$ with the vectors $(\mathcal{T}_\beta)$, $\beta \in M_2$ in a bounded set into the relation (2.16) or (5.7), respectively. Then we deduce

$$\sum_{i\in M_1} \xi_i a_i(x,\xi_\alpha,\mathcal{T}_\beta) + \sum_{i\in M_2} \mathcal{T}_i a_i(x,\xi_\alpha,\mathcal{T}_\beta) \geq$$

$$\geq c_1 \sum_{i\in M_1} \xi_i g_i(\xi_i) - c_2$$

and with respect to (2.2) we estimate

$$|\mathcal{T}_i a_i(x,\xi_\alpha,\mathcal{T}_\beta)| \leq c_3\Big(1 + \sum_{i\in M_1}|g_i(\xi_i)|\Big)$$

for each $i \in M_2$.

From these inequalities we conclude easily that (2.11a) is satisfied. The rest of the proof is the same as that of Theorem 3, § 2.

§ 7.

Applying the methods of the calculus of variation we obtain a theorem guaranteeing the existence of a weak solution for the problem (2.3) by weaker assumptions about the coercivity as in Theorem 11 and Theorem 12. A similar idea was used in my paper [2].

With regard to (2.2) and (2.4) we construct the functional (2.5), i.e.,

$$\phi(u) = \sum_{i \in M} \int_0^1 dt \int_\Omega D^i u \, a_i(x, tD^j u)\, dx - (u,f)_\Omega - (u,g)_{\partial\Omega}$$

which is continuous in the space $W^k_{\vec{G}}(\Omega)$ and has the Gateâux's differential at every point $u \in W^k_{\vec{G}}$ - see Lemma 2, § 2 and [4].

<u>Theorem 13</u>. Let the conditions (2.2),(2.4),(2.9),(2.10) and (5.7) be fulfilled. Then there exists a solution of the problem (2.3).

<u>Proof</u>. Let us look for the minimum of the functional (2.5) on the convex closed set $u_0 + V_{\vec{G}}$. First we prove the coercivity and the weak lower-semicontinuity of the functional (2.5). From (5.7),(2.9) and due to Theorem 10 we obtain

$$\lim_{\|u\|_{W^k_{\vec{G}}} \to \infty} \|u\|^{-1}_{W^k_{\vec{G}}} \int_\Omega \sum_{i \in M} D^i u \, a_i(x, D^j u)\, dx = \infty$$

and hence similarly as in Theorem 2, § 2 - see also [4] - it can be proved

(7.1) $$\lim_{\|u\|_{W^{k}_{\vec{G}}}\to\infty} \Phi(u) = \infty .$$

Now, we prove the weak lower-semicontinuity of $\Phi(u)$. Suppose that $v_n \rightharpoonup v$ with $n \to \infty$ (weak convergence) in the space $W^{k}_{\vec{G}}$.

$$\Phi(v_n) - \Phi(v) - D\Phi(v, v_n - v) =$$
$$= \int_0^1 D\Phi(v + t(v_n - v), v_n - v)\,dt - D\Phi(v, v_n - v) =$$
$$= \int_0^1 dt \int_\Omega \sum_{i\in M_1} D^i(v_n - v)\,[a_i(x, D^\alpha v + tD^\alpha(v_n - v),$$
$$D^\beta v + tD^\beta(v_n - v)) - a_i(x, D^\alpha v, D^\beta v + tD^\beta(v_n - v))]\,dx +$$
$$+ \int_0^1 dt \int_\Omega \sum_{i\in M_1} D^i(v_n - v)\,[a_i(x, D^\alpha v, D^\beta v + tD^\beta(v_n - v)) -$$
$$- a_i(x, D^\alpha v, D^\beta v)]\,dx + \int_0^1 dt \int_\Omega \sum_{i\in M_2} D^i(v_n - v)\,.$$
$$.\,[a_i(x, D^j v + tD^j(v_n - v)) -$$
$$- a_i(x, D^j v)]\,dx \equiv A_n + B_n + D_n .$$

Since $v_n \rightharpoonup v$ with $n \to \infty$, it holds $\lim_{n\to\infty} D\Phi(v, v_n - v) = 0$. Due to the assumption (2.10) it is $A_n \ge 0$. With respect to (2.9), we deduce that $D^i v_n \to D^i v$ with $n \to \infty$ in the norm of the space $L^*_{G_i}(\Omega)$ for all $i \in M_2$. In view of the fact $\|v_n\|_{W^{k}_{\vec{G}}} \le c_3$, we obtain

$\|a_i(x, D^j v + tD^j(v_n - v))\|_{P_i} \le c_4$ for each $t \in \langle 0, 1\rangle$

and $i \in M_2$ - see Lemma 1, § 2. Hence, using the Hölder's inequality, we conclude $\lim_{n\to\infty} D_n = 0$.

From (2.9) we deduce

$$a_i(x, D^\alpha v, D^\beta v + t D^\beta (v_n - v)) \to a_i(x, D^\alpha v, D^\beta v)$$

with $n \to \infty$, in the norm of the space $L^*_{p_i}(\Omega)$, uniformly with respect to $t \in \langle 0, 1 \rangle$ for all $i \in M$ - see Lemma 1, § 2. Thus, we conclude $\lim\limits_{n\to\infty} B_n = 0$ and hence the lower-semicontinuity of (2.5) with respect to weak convergence is proved.

If $\{u_n\} \in u_0 + V_{\vec{\sigma}}$ is a minimizing sequence, then $\| u_n \|_{W^k_{\vec{\sigma}}} \leq c$ in view of (7.1). Since $W^k_{\vec{\sigma}}$ is a reflexive space there exists a subsequence $\{u_{n_k}\}$ from $\{u_n\}$ so that $u_{n_k} \rightharpoonup u \in W^k_{\vec{\sigma}}$ with $k \to \infty$. The set $u_0 + V_{\vec{\sigma}}$ is weakly closed and hence $u \in u_0 + V_{\vec{\sigma}}$. Due to the weak lower-semicontinuity of the functional (2.5) we conclude that $\phi(v)$ attains its minimum on the set $u_0 + V_{\vec{\sigma}}$ at the point $u \in u_0 + V_{\vec{\sigma}}$. If $v \in V_{\vec{\sigma}}$, then $D\phi(u, v) = 0$ (Gateâux´differential at the point u) for all $v \in V_{\vec{\sigma}}$. Thus, u is a solution of the problem (2.3).

References

[1] J. KAČUR: On existence of the weak solution for non-linear partial differential equations of elliptic type, Comment.Math.Univ.Carolinae 11(1970),137-180.

[2] J. KAČUR: On boundedness of the weak solution for some class of quasilinear partial differential equations, Časopis pro pěst. matematiky, to appear.

[3] T.K. DONALDSON and Neil S. TRUDINGER: Orlicz-Sobolev spaces and imbedding theorems, Macquarie University,

1970.

[4] J. NEČAS: Les équations elliptiques non linéaires, (L'école d'été, Tchécoslovaquie 1967), Czech. Math. Journal 2(1969), 252-274.

Vilová 15,
Bratislava,
Československo

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