Commentationes Mathematicae Universitatis Carolinae
13,2 (1972)

ON DENSE SUBSPACES OF CERTAIN TOPOLOGICAL SPACES

G.M. REED, Athens

In this paper, the following results are obtained: (i) Each stratifiable space in which each point has a σ-closure preserving local base has a dense subspace which is an M_1-space. (ii) There exists a paracompact σ-space (due to R.W. Heath) which has no dense stratifiable subspace. (iii) Each semi-stratifiable space has a dense subspace which is a σ-space.

I. Introduction. Consider the following relationships between certain widely studied abstract spaces (all spaces are to be T_1). (1) Each M_1-space is a stratifiable space (M_3-space) [3]. (2) Each stratifiable space is a paracompact σ-space [6]. (3) Each σ-space is a semi-stratifiable space [4]. The converses of statements (2) and (3) are shown to be false in [5] and [1] respectively and the validity of the converse of statement (1) is an open question. However, it follows from the author's results in [9], that in each of the statements (1),(2), and (3) a first countable space of the second type contains a dense subspace

---

AMS, Primary: 54E20, 54B05 Ref.Ž. 3.961.3

which is of the first type. It is the purpose of this paper to investigate this relationship for non-first countable spaces.

II. Preliminaries.

Notation 1.1. If M is a subset of the space S, then $CL(M)$ will denote the closure of M in S. If H is a set collection, then H^* will denote the union of the members of H.

Definition 1.2. A collection G of subsets of the space S is said to be closure preserving provided that for each subcollection H of G, $CL(H^*) = \{CL(h) \mid h \in H\}^*$. A collection G of subsets of the space S is said to be σ-closure preserving if it is the union of countably many closure preserving collections.

Definition 1.3. A collection G of subsets of the space S is said to be a network for S provided that if $p \in S$ and D is an open set containing p, then there exists an element g of G such that $p \in g$ and $g \subset D$.

Definition 1.4. [3] An M_1-space is a regular space having a σ-closure preserving base.

Definition 1.5. [3] An M_2-space is a regular space S having a σ-closure preserving quasi-base.

Definition 1.6. [2] A space X is a stratifiable space (M_3-space) if to each open $U \subset X$, one can assign a sequence $U_1, U_2, \ldots$ of open subsets of X such that

(a) $CL(U_n) \subset U$ for each n,

(b) $\bigcup U_n = U$,

(c) $\mathcal{U}_n \subset \mathcal{V}_n$, whenever $\mathcal{U} \subset \mathcal{V}$.

Definition 1.7. [7] A σ-space is a space X having a σ-locally finite network.

Definition 1.8. (Due to E.A. Michael.) A space X is semi-stratifiable if to each open $\mathcal{U} \subset X$, one can assign a sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of open subsets of X which satisfy (b) and (c) of Definition 1.6.

Lemma 1.9. [4] A necessary and sufficient condition for a space X to be semi-stratifiable is that for each $x \in X$, there exists a sequence $g_1(x), g_2(x), \ldots$ of open subsets of X such that (i) $\bigcap g_i(x) = x$ and (ii) if $y \in X$ and $x_1, x_2, \ldots$ is a sequence of points in X such that for each i, $y \in g_i(x_i)$, then $x_1, x_2, \ldots$ converges to y .

III. Theorems. The author has not been able to decide whether each non-first countable stratifiable space has a dense subspace which is an M_1-space. However, the following theorem is a partial answer. By the statement that G is a local base for the point p of the space S is meant that G is a collection of open subsets of S such that if p is contained in the open set D , then there exists an element g of G such that $p \in g$ and $g \subset D$.

Theorem 3.1. Each stratifiable space in which each point has a σ-closure preserving local base has a dense subspace which is an M_1-space.

Proof. Let S be a stratifiable space and for each poin p of S let $B(p)$ denote a σ-closure preserving local base for p .

Each stratifiable space is a σ-space [6]. Thus, let $H = \bigcup H_i$ denote a network for S where for each i, H_i is locally finite. For each i, let K_i denote a point set containing one point from each element of H_i and note that K_i is discrete in S. Since each stratifiable space is paracompact [3] and hence collectionwise normal, for each i, there exists a discrete collection G_i of open sets in S covering K_i such that each element of G_i contains only one point of K_i. For each i and each point p of K_i, let $V_i(p) = \{ b \in B_p \mid b$ is contained in the element of G_i which contains $p \}$. Note that $V_i(p) = \bigcup_j V_{i,j}(p)$ where for each j, $V_{i,j}(p)$ is closure preserving.

Now, let $K = \bigcup K_i$ and for each i and j, let $V_{i,j} = \{ b \cap K \mid b \in V_{i,j}(p)$ and $p \in K_i \}$. It follows that K is a dense subset of S and $\bigcup_i \bigcup_j V_{i,j}$ is a σ-closure preserving base for K, regarded as space. Thus, K is an M_1-space.

<u>Theorem 3.2.</u> There exists a paracompact σ-space which has no dense stratifiable subspace.

<u>Proof.</u> In [7], Heath gave an example of a regular, countable space X which is not stratifiable. Since X is a paracompact σ-space, it suffices to show that X also has no dense stratifiable subspace.

The space X in [7] is based on the existence of a collection $\mathcal{F}'$ of subsets of N, the set of all natural numbers, such that (1) $\mathcal{F}'$ has c members, (2) for any choice of $n + m$ distinct members, $F_1, F_2, \dots, F_n, F_{n+1}, \dots$

$\ldots, F_{n+m}$ of $\mathcal{F}'$, $F_1 \cap F_2 \cap \ldots \cap F_n \cap$
$\cap (N - F_{n+1}) \cap (N - F_{n+2}) \cap \ldots \cap (N - F_{n+m}) \neq \emptyset$
and (3) for any two natural numbers x and y, there is a member of $\mathcal{F}'$ that contains exactly one of x and y. The points of X are the points of N and $\mathcal{F} = \mathcal{F}' \cup$
$\cup \{N - F \mid F \in \mathcal{F}'\}$ is a subbasis for the topology of X.

Now, suppose that S is a dense subspace of X. Since for each element F of $\mathcal{F}'$, both F and $(N - F)$ are open in X, $F \cap S \neq \emptyset$ and $(N - F) \cap S \neq \emptyset$. Thus let $\mathcal{G}' = \{F \cap S \mid F \in \mathcal{F}'\}$. It follows that the collection $\mathcal{G}'$ has properties (1),(2),and (3) above with respect to the subset S of N and that $\mathcal{G} = \mathcal{G}' \cup \{S - G \mid G \in \mathcal{G}'\}$ is a subbasis for S. Replacing N by S, $\mathcal{F}'$ by $\mathcal{G}'$, and $\mathcal{F}$ by $\mathcal{G}$, one can use the same argument given by Heath to show that S is also not stratifiable.

The proof given for the following theorem is a modification of the proof given in [9] for the existence of a dense developable subspace in a semi-metric space.

Theorem 3.3. Each semi-stratifiable space S has a dense subspace which is a σ-space.

Proof. It is sufficient to show that S has a dense subspace X which is the union of countably many subsets each of which is discrete in X.

For each point p of S, let $g_1(p), g_2(p), \ldots$ be a sequence of open sets in S as in Lemma 1.9. Denote by Ω a well-ordering of the points of S. For each j, let X_j

be the subset of S such that: (1) the first element of K_j is the first element of S with respect of Ω. (2) If I is an initial segment of K_j, then the first element p of $K_j - I$ is the first element of S with respect to Ω such that p is not a limit point of I and p is not in $g_j(q)$ for q in I. (3) If K'_j is a subset of S having properties (1) and (2) then either K'_j is K_j or K'_j is an initial segment of K_j.

It follows that $K = \bigcup K_i$ is dense in S. For suppose that $p \in S - CL(K)$. If for each i, $g_i(k_i)$ contains p for some point k_i in K, then the sequence $k_1, k_2, \dots$ would converge to p and p would be in $CL(K)$. Thus for some j, there exists no element k of K_j such that $g_j(k)$ contains p. But if this were true, p would be in K_j and hence in K. This contradicts the choice of p.

Now, let $X_1 = K_1$ and for each $i > 1$, let $X_i = K_i - (CL(\bigcup_{j=1}^{i-1} X_j) \cap K_i)$. It follows that $X = \bigcup X_i$ is dense in S. Consider X_i for each i. By the construction of K_i, no point of X_i is a limit point of X_i. And by the construction of X_i, no point of $\bigcup_{j=i+1}^{\infty} X_j$ is a limit point of X_i. Thus if X_i has a limit point q in X, q must be in $\bigcup_{j=1}^{i-1} X_j$. But for each p in X_i, there exists an m such that p is not in $g_m(q)$ for q in $\bigcup_{j=1}^{i-1} X_j$. If this were not true, the sequence $q_1, q_2, \dots$ where for each m, q_m is in $\bigcup_{j=1}^{i-1} X_j$ and p is in $g_m(q_m)$, would converge to p and hence p would be a limit point of $\bigcup_{j=1}^{i-1} X_j$. Thus for each n , let $X_{i,n} = \{p \text{ in } X_i \mid p$ is not in $g_n(q)$ for q in $\bigcup_{j=1}^{i-1} X_j\}$. Note for each n, $X_{i,n}$ has no limit point in X .

Thus $X = \bigcup_i \bigcup_n X_{i,n}$ is a dense subspace of S which is the union of countably many subsets each of which is discrete in X .

References

[1] E.S. BERNEY: A regular Lindelöf semi-metric space which has no countable network, Proc.Amer.Math.Soc.26 (1970),361-364.

[2] C.J.R. BORGES: On stratifiable spaces, Pacific J.Math. 17(1966),1-16.

[3] J.G. CEDER: Some generalizations of metric spaces, Pacific J.Math.11(1961),105-125.

[4] G.C. CREEDE: Concerning semi-stratifiable spaces, Pacific J.Math. 32(1970),47-54.

[5] R.W. HEATH: A paracompact semi-metric space which is not an M_3-space, Proc.Amer.Math.Soc.17(1966),868-870.

[6] - : Stratifiable spaces are σ-spaces, to appear.

[7] - : On a non-stratifiable countable space, Proc. of the Emory Univ.Topology Conf.(1970),119-122.

[8] A. OKUYAMA: σ-spaces and closed mappings, I, Proc.Japan Acad.44(1968),472-477.

[9] G.M. REED: Concerning first countable spaces, Fund.Math., to appear.

Auburn University and
Ohio University

(Oblatum 28.1.1972)