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UPPER BOUND FOR THE NUMBER OF EIGENVALUES FOR NONLINEAR OPERATORS

(Preliminary communication)

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<u>Introduction</u>. Let f and g be two nonlinear functionals defined on a real Hilbert space R. We consider the eigenvalue problem

$$(1)\qquad \begin{cases} \lambda f'(u) = g'(u) \\ f(u) = \varphi \end{cases}$$

($\varphi > 0$ is a prescribed number, f' and g' denote Fréchet derivatives of f and g respectively).

Under some assumptions on f and g it is known that there exist an infinite number of points $u \in R$ and infinite $\lambda \in E_1$ satisfying (1)(see [2], [3],[4]). Such a theorem was first obtained by L.A. Ljusternik and L. Schnirelman in 1935 - 1939.

In this preliminary note we give abstract theorems with

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reasonable assumptions on the functionals f and g about the result concerning upper bound for the number of λ's and u's solving the eigenvalue problem (1) and the application to the differential and integral equations.

Abstract theorems. Let R be a real Hilbert space.

Theorem 1. Let f and g be two real analytic functionals on R in the sense of [1], $a > 0$, $b > 0$. Suppose

(2) $f(tu) = t^a f(u)$ for $t > 0$ and $u \in R$,

(3) $g(tu) = t^b g(u)$ for $t > 0$ and $u \in R$,

(4) there exists $c_1 > 0$ such that $f(u) \geq c_1 . \|u\|^a$ for each $u \in R$,

(5) there exists $c_2 > 0$ such that $d^2 f(u, h, h) \geq$ $\geq c_2 \|h\|^2 . \|u\|^{a-2}$ for each $u, h \in R$,

(6) g' is a completely continuous mapping from R to R.

Then the eigenvalue problem (1) has a solution only for finite or countable infinite λ's and only one possible cummulation point of these λ's is zero.

Theorem 2 (special case). Let f be a scalar product in R (generally the theorem is true if $\{u \in R;\ f(u) = \varphi\}$ is a "real-analytic manifold") and g be a real analytic functional on R satisfying the relation (5) and suppose that

(7) $$g(u) \neq 0 \Longrightarrow g'(u) \neq \theta \ .$$

Denote by $\mathcal{U}$ the set of u's for which the eigenvalue problem (1) has a solution.

Then the set $g(\mathcal{U}) \cap \langle \varepsilon, \infty)$ is a finite set for each $\varepsilon > 0$. (The point $\gamma \in g(\mathcal{U})$ is called a critical number for the eigenvalue problem (1).)

Remark. Suppose, moreover, in Theorem 1 that

(8) f and g are even functionals,

(9) f' and g' are bounded operators,

(10) $u \in R \Longrightarrow g(u) \geq 0, \ g(u) = 0 \Longleftrightarrow u = \theta$,

(11) f' and g' are uniformly continuous on each bounded set.

Then there exists a sequence $\{\lambda_n\}_{n=1}^{\infty}$, $\lambda_n \longrightarrow 0$, $\lambda_n > 0$ such that only for $\lambda = \lambda_n$ the eigenvalue problem (1) has a solution and if $a = b$ for $\lambda \notin \{\lambda_n\}_{n=1}^{\infty} \cup \{0\}$ the operator $A_\lambda = \lambda f' - g'$ maps R onto R .

Applications

Example 1. We consider the Lichtenstein integral equation

$$\lambda u(s) = \sum_{m=1}^{\infty} \int_0^1 \dots \int_0^1 K_m(s, t_1, \dots, t_m) u(t_1) \dots u(t_m) \, dt_1 \dots dt_m$$

for $u \in L_2 \langle 0, 1 \rangle$ under the same assumptions as in [2]. Then the assumptions of Theorem 2 are fulfilled.

Example 2. The degenerated Lichtenstein integral equation

$$\lambda u(s) = \int_0^1 \dots \int_0^1 K_m(s, t_1, \dots, t_m) u(t_1) \dots u(t_m) \, dt_1 \dots dt_m$$

under the same assumptions on the function K_m as in Example 1 satisfies the conditions in Theorem 1. Analogously for the equation

$$\lambda \langle u, u \rangle^{n} u(s) = \int_0^1 \dots \int_0^1 K_m(s, t_1, \dots, t_m) u(t_1) \dots u(t_m) \, dt_1 \dots dt_m$$

where $\langle u, u \rangle$ is a scalar product in $L_2 \langle 0, 1 \rangle$.

Example 3. Let $\Omega \subset E_n$ be a bounded domain and we consider the weak solution of the Dirichlet boundary value problem for the equation

$$\begin{cases} \lambda (-1)^{m+1} \Delta^m u + g(u) = 0 \\ \qquad D^\alpha u = 0 \quad \text{on boundary, } |\alpha| \le m - 1 . \end{cases}$$

If $2m < n$ we suppose that g is a polynomial function of the degree $k < \frac{n + 2m}{n - 2m}$. Then the assumptions of Theorem 1 or Theorem 2 are satisfied. The same problem can be solved on the base of our abstract theorems in the case $2m \ge n$, too.

The proofs and a detailed study of examples will appear later in Ann.Scuola Norm.Sup.Pisa.

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