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# MIXED PROBLEM FOR SEMILINEAR HYPERBOLIC EQUATION OF SECOND ORDER WITH THE DIRICHLET BOUNDARY CONDITION

Preliminary communication

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The following mixed problem is considered in the auhor's prepared paper [3] : Let

$$L = \frac{\partial^2}{\partial t^2} + \sum_{i=1}^{n} h_i(x,t)\frac{\partial^2}{\partial x_i \partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial}{\partial x_j}\right) +$$

+ first order

be a linear operator of hyperbolic type, i.e. the condition

$$a_{ij} = \bar{a}_{ji} \; ; \sum_{i,j=1}^{n} a_{ij}(x,t) z_i \bar{z}_j \ge \delta |z|^2, \; z \in C^n, \; \delta > 0$$

holds in the definition domain $Q \equiv \Omega \times (0,T)$ of L ( $\Omega \subset R^n$ is a bounded domain, $0 < T < \infty$ ) and let $h_i$ be real-valued functions. It is required to find a function $u \in C(0,T; H^k) \equiv$

$$\equiv \bigcap_{i=0}^{k} C^{(i)}(0,T; W_2^{(k-i)}(\Omega)) \; , \quad k \ge 2 \quad ,$$

satisfying the equation

$$(1) \quad Lu = f(x,t,u(x,t), u'(x,t), \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}) + h(x,t)$$

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in $Q$ ($u' = \frac{\partial u}{\partial t}$) , the initial conditions

(2) $$u(0) = u_0 , \quad u'(0) = u_1$$

in $\Omega$ and the Dirichlet boundary condition in the sense

(3) $$u - g \in C(0,T;\mathring{H}^k) \equiv C(0,T;H^k) \cap C(0,T;\mathring{W}_2^{(1)}(\Omega)) .$$

By means of successive approximations one can prove a local existence theorem:

<u>Theorem A</u>. Be $k \geq [n/2] + 2$ an integer, $\partial\Omega \in C^{(k+1),1}$ , and let the coefficients of $L$ be of the class $C^{(k-1)}(\overline{Q})$ . Be

$$u_0 \in W_2^{(k)}(\Omega), \quad u_1 \in W_2^{(k-1)}(\Omega) ,$$

$$h \in C(0,T;H^{k-2}) \cap C^{(k-1)}(0,T;L_2(\Omega)) ,$$

$$g \in C(0,T;H^k), \quad g' \in C(0,T;H^k)$$

and let $f(x,t,z_1,\dots,z_{n+2}) \in C^{(k-1)}(\overline{Q} \times C^{n+2})$, $D^{k-1}f$ be locally $\lambda$-Hölder continuous in the variables $z_1,\dots,z_{n+2}$ for some $\lambda \in (0,1\rangle$ . Assume further that the necessary compatibility conditions hold.

Then there exists $\Delta \in (0,T\rangle$ such that our mixed semi-linear problem (1) - (3) has on $\langle 0,\Delta\rangle$ a unique solution $u \in C(0,\Delta;H^k)$ .

Then a question of a global solution is considered using an apriori estimate:

Definition. We say that an apriori estimate for the semi-linear mixed problem (1) - (3) holds, if

$\exists C_A \geq 0 \; \forall t \in (0,T) : u \in C(0,t;H^k)$ is a solution of (1) - (3) $\Longrightarrow$

$$\Longrightarrow \sum_{i=0}^{[n/2]+2} \| u^{(k-i)}(s) \|_{W_2^{(i)}(\Omega)} \leq C_A \quad \forall s \in \langle 0, t \rangle .$$

A global solution of the problem is found by continuation of the known local solution from Theorem A.

Theorem B. Let the assumptions of Theorem A be satisfied and, moreover, let an apriori estimate hold.

Then there exists a unique solution $u \in C(0,T;H^k)$ of the mixed problem (1) - (3) on the whole interval $\langle 0, T \rangle$ .

Remark: If our non-linear term does not depend on derivatives of $u$ , then Theorems A,B hold for $k = [n/2]+1$, too.

In the last paragraph of the mentioned paper some sufficient conditions for the existence of apriori estimate are given, mainly:

Theorem C. Let f be bounded in $\overline{Q} \times C^{n+2}$ together with all derivatives up to the order $[n/2]+1$ . Then the apriori estimate holds.

Theorem D. Be $g = 0$ and let the assumptions of Theorem A be satisfied. Let for $u \in C(0,t;\overset{\circ}{H}{}^2)$ , $t \in (0,T\rangle$ ,

$$Lu = f(x, s, u(x, s)),\ u(0) = u_0,\ u'(0) = u_1 .$$

Let us suppose that there exists a real-valued function $F(x, t, z)$ defined on $\bar{Q} \times C$ such that $\partial F/\partial(\mathrm{Re}\, z) = \mathrm{Re}\, f$, $\partial F/\partial(\mathrm{Im}\, z) = \mathrm{Im}\, f$, $F \leq C_F$, $(C_F \geq 0)$, and either $-\partial F/\partial t \leq C_F'(C_F - F)$ or $|\partial F/\partial t| \leq$ $\leq C_F'(1 + |z|^2)$, $C_F' \geq 0$.

Then there exists a constant $C_1 > 0$ such that

(4) $\|u(s)\|_{W_2^{(1)}(\Omega)} + \|u'(s)\|_{L_2(\Omega)} \leq C_1 \quad \forall s \in \langle 0, t \rangle$

and consequently apriori estimate in case $n = 1$ holds.

<u>Theorem B</u>. Let the assumptions of Theorem A be satisfied and let $u \in C(0, t; H^2)$, $t \in (0, T\rangle$, be such a solution of (1) - (3) that (4) holds. Let the function $f(x, t, z)$ further satisfy

$$\left|\frac{\partial f}{\partial t}\right| \leq C_f (1 + |z|^{a+1}) ,$$

$$\left|\frac{\partial f}{\partial z}\right| \leq C_f (1 + |z|^{a})$$

where $a = 2/n - 2$ for $n > 2$, $0 \leq a < \infty$ for $n \leq 2$, $C_f \geq 0$.

Then there exists a constant $C_2 > 0$ such that

$$\sum_{i=0}^{2} \|u^{(2-i)}(s)\|_{W_2^{(i)}(\Omega)} \leq C_2 \quad \forall s \in \langle 0, t \rangle$$

and consequently apriori estimate holds for $n = 2$, $n = 3$.

Finally it is shown in examples that the results of J. Sather from [1],[2] are included as a particular case.

R e f e r e n c e s

[1] J. SATHER: The initial-boundary value problem for a non-linear hyperbolic equation in relativistic quantum mechanics, J.Math.Mech.16(1966),27-50.

[2] J. SATHER: The existence of a global classical solution of the initial boundary value problem for $\Box u + u^3 = f$, Arch.Rat.Mech.Anal.22(1966), 292-307.

[3] A. DOKTOR: Mixed problem for semilinear hyperbolic equation of second order with the Dirichlet boundary condition. To appear in Czech.Math.J.

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