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SURJECTIVITY AND FIXED POINT THEOREMS
(Preliminary communication)

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Let X be a LCS (Hausdorff locally convex space), C a closed convex subset of X, $exp\ C$ the set of all subsets of C and A a partially ordered set such that: $\forall a, b \in A\ \exists\ max\{a, b\} \in A$. A mapping μ : $: exp\ C \to A$ is said to be a mnc (measure of noncompactness) on C if $\mu(\overline{co}\ M) = \mu(M)$ for all $M \in exp\ C$. Consider the following conditions on a mnc μ on C :
(1) $M \subseteq N \subseteq C$ implies $\mu(M) \leq \mu(N)$;
(2) $M, N \in exp\ C$ implies $\mu(M \cup N) = max\{\mu(M), \mu(N)\}$;
(3) $M \in exp\ C$ implies $\mu(-M) = \mu(M)$ (for C symmetric); (4) $M \in exp\ C$ implies $\mu(\{0\} \cup M) = \mu(M)$ (for C containing 0); (5) $x \in C$ and $M \in exp\ C$ together imply $\mu(x + M) = \mu(M)$ (for C a cone).

On any NLS (normed linear space) X there are two natural mnc's χ_X and α_X defined by $\chi_X(M) = inf\{\varepsilon > 0 : M$ can be covered by a finite number of ε-balls$\}$, $\alpha_X(M) = inf\{\varepsilon > 0 : M$ has a finite ε-covering$\}$ (here $A = [0, +\infty]$).

Let $F : C \to X$ be a continuous mapping and μ a

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mnc on $\overline{co}\,(C \cup F(C))$. We shall write $F \in \mathcal{D}(\mu) = \mathcal{D}(\mu, C)$ if $M \subseteq C$ and $\mu(F(M)) \geq \mu(M)$ together imply that M is relatively compact.

__Theorem 1.__ Let X be a LCS, $0 \in C$ an open subset of X, $F : \overline{C} \longrightarrow X$ a mapping such that $F \in \mathcal{D}(\mu, \overline{C})$ where μ is a mnc on $\overline{co}\,(C \cup F(C))$ satisfying Conditions (1) and (4). If $Fx \neq tx$ for all $x \in \partial C$ (= the boundary of C) and all $t > 1$, then F has a fixed point in $\overline{C}$.

__Theorem 2.__ Let X be a NLS, μ a mnc defined on bounded subsets of X and satisfying Conditions (2),(3) and (5). Let $\{C_n\}_{n=1}^{\infty}$ be a sequence of open, symmetric, strictly starshaped (i.e., $[0,1)x \subseteq C_n$ for each $x \in \partial C_n$) subsets of X such that $dist\,(0, \partial C_n) \longrightarrow \infty$. Let $F : X \longrightarrow X$ be a mapping such that $F \in \mathcal{D}(\mu)$, $\|\Phi(x)\| \to \infty$ as $\|x\| \longrightarrow \infty$, $x \in \bigcup_{n=1}^{\infty} \partial C_n$. Suppose that $\Phi(-x) \neq t\Phi(x)$ for all $x \in \bigcup_{n=1}^{\infty} \partial C_n$ and all $t > 0$. (Here $\Phi = I - F$.) Then $I - F$ is surjective.

__Corollary 1.__ Let X be a NLS and C, F, μ as in Theorem 1. Suppose that for each $x \in \partial C$ there is a function $\varphi_x : [0, +\infty] \longrightarrow [0, +\infty]$ such that $a, b > 0$ implies $\varphi_x(a+b) > \varphi_x(a) + \varphi_x(b)$. If $\varphi_x(\|Fx\|) \leq \varphi_x(\|x\|) + \varphi_x(\|x - Fx\|)$ for each $x \in \partial C$, then F has a fixed point in $\overline{C}$.

Corollary 2. Let X, C, F, μ be as in Theorem 1. Suppose that $0 \in C$ and that C is strictly starshaped. If $F(\partial C) \subseteq \bar{C}$, then F has a fixed point in $\bar{C}$.

Corollary 3. Let X be a NLS, μ a mnc on bounded subsets of X satisfying Conditions (1),(4) and (5), $F: X \longrightarrow X$ a mapping such that $F \in \mathcal{D}(\mu)$. Let $\{C_m\}_{m=1}^{\infty}$ be a sequence of open subsets of X containing 0 and $\{a_m\}_{m=1}^{\infty}$ a positive sequence tending to $+\infty$ as $m \longrightarrow +\infty$, such that $\|Fx\| \leq \|x\| - a_m$ for each $x \in \partial C_m$ $(m \geq 1)$. Then $I - F$ is surjective.

Corollary 4. Let X be a NLS, μ a mnc as in Theorem 2, $F: X \longrightarrow X$ a mapping with $F \in \mathcal{D}(\mu)$. Suppose that F has an asymptotic derivative $F'(\infty)$ such that $I - F'(\infty)$ is an (topological) isomorphism of X. Then $I - F$ is surjective.

Remarks. 1. Analogous results hold for mappings of the form $T - S$.

2. Some results of [3] and [4](and [1]) can (and will) be proved for mappings of this type.

3. For some mnc's μ, if $F: X \longrightarrow X$ (X a NLS) is in a certain subclass of $\mathcal{D}(\mu)$ and has an asymptotic derivative $F'(\infty)$, then $F'(\infty) \in \mathcal{D}(\mu)$.

4. Some mnc's induce, in a natural way, the mnc's on factor spaces.

5. If X is a NLS and $\delta_X^{\sim}(\varepsilon) = \sup \{\|\frac{x+y}{2}\|$:

$: x, y \in X, \|x-y\| \geq \varepsilon, \|x\|, \|y\| \leq 1 \}$,

then $\frac{1}{2} \alpha_X \leq \chi_X \leq \sigma_X(1) \cdot \alpha_X \leq \alpha_X$.

A detailed study of these problems including complete references and applications to nonlinear integral and differential equations will be given in subsequent papers.

References

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