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ON THE EXISTENCE OF SCHAUDER BASES IN SOBOLEV SPACES

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1. Introduction. Let Ω be a domain in E_N. We denote, as usually, $\alpha = (\alpha_1, \dots, \alpha_N)$, $\alpha_i \ge 0$, integer, $i = 1, 2, \dots, N$ and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_N^{\alpha_N}}$$

where $|\alpha| = \sum_{i=1}^{N} \alpha_i$. We define the Sobolev space $W_p^k(\Omega)$ (for $k \ge 0$, integer, $p \ge 1$) as a subspace of $L_p(\Omega)$ consisting of all functions f for which $D^\alpha f \in L_p(\Omega)$ if only $|\alpha| \le k$, normed by

$$\|f\|_{W_p^k} \overset{def}{=} \Big(\sum_{|\alpha| \le k} \int_\Omega |D^\alpha f|^p \, dx \Big)^{\frac{1}{p}}$$

($D^\alpha f$ means the derivative in the sense of distributions.)

Further, denoting by $\mathcal{D}(\Omega)$ the set of all infinitely differentiable functions on Ω with compact supports in Ω, we define $\overset{\circ}{W}{}_p^k(\Omega)$ as a closure of $\mathcal{D}(\Omega)$ in $W_p^k(\Omega)$.

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In many papers the existence of Schauder bases of Sobolev spaces (and their subspaces) is desired for the purpose of the demonstration of the existence of solutions of non-linear boundary value problems. There was proved in [3] that for $\Omega \subset E_1$ the spaces $W_p^k(\Omega)$ and $\mathring{W}_p^1(\Omega)$ have the Schauder bases. Their elements are constructed as primitive functions of elements of the Haar's orthogonal system.

The existence of Schauder bases of $W_p^k(E_N)$ and $\mathring{W}_p^k(E_N)$ $(p > 1)$ follows immediately from the assertion that $W_p^k(E_N)$ is isomorphic with $L_p(E_N)$ (see [13]) and $\mathring{W}_p^k(E_N) = W_p^k(E_N)$ (see [12]).

Let us note that before we have been able to prove the existence of Schauder bases of $W_p^k(\Omega)$, $\mathring{W}_p^k(\Omega)$ (Ω - bounded with $\partial\Omega$ sufficiently smooth), we defined the concept of the "space with the usual structure" ([5]). This concept which is closely connected with the existence of Schauder basis has been useful for some considerations in the existence theory of non-linear boundary value problems. At the same time we proved that for $\partial\Omega$ sufficiently smooth both $W_p^k(\Omega)$ and $\mathring{W}_p^k(\Omega)$ are the spaces with the usual structure.

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ing the result on which the proof of the existence of Schauder bases in Sobolev spaces is based.

In Section 2 we give the definitions of the spaces with the usual structure, Schauder basis and $(\sigma)_k$ -property and we study the connections between spaces with these properties. In Section 3 we give the proof of the assertion that for Ω bounded, $\partial\Omega$ sufficiently smooth, the spaces $W_p^k(\Omega)$, $\mathring{W}_p^k(\Omega)$ (and some other spaces) have Schauder bases. Finally, in Section 4, some open problems are formulated.

2. Banach spaces with the usual structure

In the sequel we will use the following properties of Banach spaces:

Definition 1. A Banach space X is said to have the usual structure if there exists a sequence $\{P_n\}$ of operators of X into X such that

(i) P_n is continuous $(n = 1, 2, \dots)$,

(ii) $P_n(-x) = -P_n(x)$ for each $x \in X$ and all n,

(iii) $P_n(X)$ is contained in some finite dimensional subspace of X,

(iv) $x_n \rightharpoonup x \Rightarrow P_n x_n \rightharpoonup x$ for each $\{x_n\} \subset X$ and all $x \in X$ (the symbol " $\rightharpoonup$ " denotes the weak convergence in X).

Definition 2. A Banach space X is said to have the linear usual structure if there exists a sequence $\{P_n\}$ of linear operators of X into X such that (i) - (iv) from the definition 1 are valid.

Definition 3. Let $k \geq 1$. A Banach space X is said to have a property $(\pi)_k$ if there exists a sequence $\{X_n\}$ of finite dimensional subspaces of X and a sequence $\{P_n\}$ of linear projectors (i.e. linear operators such that $P_n^2 = P_n$) from X onto X_n for each n such that the following conditions are valid:

(i) $\quad X_n \subset X_{n+1} \quad (n = 1, 2, \ldots)$,

(ii) $\quad \overline{\bigcup_{n=1}^{\infty} X_n} = X$,

(iii) $\quad \| P_n x \| \leq k \| x \|$ for each $n = 1, 2, \ldots$ and all $x \in X$.

Definition 4. A sequence $\{e_n\}$ of elements of a real Banach space is called a Schauder basis if each $x \in X$ has a unique representation $x = \sum_{i=1}^{\infty} a_i e_i$, where a_i are real numbers.

Proposition 1. A Banach space with the usual structure is separable.

Proof. For arbitrary integer n denote by X_n a finite dimensional subspace of X such that $P_n(X) \subset X_n$. Condition (iv) implies that $\overline{\bigcup_{n=1}^{\infty} X_n}$ is weakly dense in X and it is obviously separable. But $\overline{\bigcup_{n=1}^{\infty} X_n}$ is a closed convex subset of X and thus it is weakly closed (see [1], Chapt.V, § 3). Thus $\overline{\bigcup_{n=1}^{\infty} X_n} = X$ and the proposition is proved.

Proposition 2. Let X be a reflexive Banach space with the usual structure. Then any sequence $\{P_n\}$ from Definition 1 is bounded, i.e. for each bounded set $M \subset X$ there exists $K > 0$ such that $\|P_n x\| \leq K$ for each integer n and all $x \in M$.

Proof. Let there exist a bounded sequence $\{x_n\} \subset X$ such that $\{\|P_n x_n\|\}$ is unbounded. Then for some $\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$ it is $x_{n_k} \rightarrow x_0 \in X$ and $\|P_{n_k} x_{n_k}\| \rightarrow \infty$, Set

$$y_n = \begin{cases} x_n & \text{for } n = n_k \\ x_0 & \text{in other case.} \end{cases}$$

Then $y_n \rightarrow x_0$ and thus (iv), and $P_n y_n \rightarrow x_0$ and therefore $\{\|P_n y_n\|\}$ is a bounded sequence. This is a contradiction.

In the next propositions we shall give the connection between various properties of Banach spaces. It is obvious that if any Banach space X has a Schauder basis $\{e_n\}$ then the operators $\{P_n\}$ defined by $P_n(\sum_{i=1}^{\infty} a_i e_i) = \sum_{i=1}^{n} a_i e_i$ are bounded and X has the property $(\pi)_k$. The existence of a Schauder basis follows under certain conditions from the property $(\pi)_k$ (for instance $k = 1$ and $\dim X_n = n$ - see [11] - or $k > 1$, $\dim X_n = n$ and $P_n \circ P_{n+1} = P_{n+1} \circ P_n = P_n$).

Proposition 3. A reflexive Banach space X with a property $(\pi)_k$ has a linear usual structure if and only if $P_n^* x^* \longrightarrow x^*$ for each $x^* \in X^*$ (P_n^* denotes the adjoint operator to P_n and X^* the adjoint space). (For a proof see [4].)

Proposition 4. A reflexive Banach space with a Schauder basis has a linear usual structure. (For proof see [4], Theorem 4 and Proposition 2 .)

Definition 5. The Banach spaces X and Y are called isomorphic if there exists a bounded linear operator T from X onto Y .

A closed linear subspace Y of a Banach space X is said to be a complemented subspace of X if there exists a linear bounded projection from X onto Y .

Proposition 5. Let X be a Banach space with the usual structure and Y its complemented subspace. Then Y is a Banach space with the usual structure. (The proof is obvious.)

Corollary 1. Complemented subspace of a reflexive Banach space with a Schauder basis has a linear usual structure.

3. Bases in some spaces connected with the boundary elliptic problem

Definition 6 (see [8]). Let $1 \leq p \leq \infty$. We denote by ℓ_p^n the space of all n-tuples $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ of numbers with the norm

$$\|\alpha\| = \Big(\sum_{i=1}^{n} |\alpha_i|^p\Big)^{\frac{1}{p}} \quad \text{if } 1 \leq p < \infty$$

and $\|\alpha\| = \max\limits_{1 \leq i \leq n} |\alpha_i|$ if $p = \infty$.

Let $1 \leq \lambda < \infty$. A Banach space X is said to be an $\mathcal{L}_{p,\lambda}$ space if for every finite dimensional subspace B of X there is a finite dimensional subspace C of X such that

(i) $B \subset C$,

(ii) $dist\,(C, \ell_p^n) \leq \lambda$, where $n = dim\, C$ and we define $dist\,(C, \ell_m^p)$ as the $inf\,(\|T\|\,\|T^{-1}\|)$, the inf is taken over all invertible operators T from C onto ℓ_m^p.

A Banach space is said to be an $\mathcal{L}_p$ space, $1 \le p \le \infty$, if it is an $\mathcal{L}_{p,\lambda}$ space for some $\lambda \in \langle 1, \infty)$.

Proposition 6 (see [9, Theorem 2.1]). Each complemented subspace of $L_p(0,1)$ which is not isomorphic to a Hilbert space, is an $\mathcal{L}_p$ space.

Proposition 7 (see [6,Theorem 5.1]). Each separable space has a Schauder basis.

The next proposition based on the results of the theory of partial differential equations gives us the possibility to judge whether Sobolev spaces and some of their subspaces are isomorphic with some $\mathcal{L}_p$ space. First, following [15] , we give some necessary notations and definitions. In the following we still suppose $p \in (1, \infty)$.

Let Ω be a bounded domain, $\Omega \subset E_N$ with the boundary $\partial\Omega \in C^\infty$ (this assumption can be properly weakened). We define

$$(1) \quad [u, v] = \int_\Omega \sum_{|\alpha|,|\beta| \le k} a_{\alpha\beta}(x)\, D^\alpha u\, \overline{D^\beta v}\, dx$$

a bilinear form with the coefficients of $C^\infty(\overline{\Omega})$.

Let for some h, $0 \le h \le k$, $C_1, C_2, \dots, C_h$ be a normal set of boundary differential operators of the order $< k$. (It means that their orders are distinct and $\partial\Omega$ is not characteristic to any of them at any point. In the case $h = 0$ we suppose the set of operators being empty.)

Set

(2) $\mathcal{U}_h = \{u \in C^\infty(\bar{\Omega}) \mid C_j u = 0$ on $\partial\Omega$, $j = 1, 2, \dots, h\}$.

(In the case $h = 0$ we define $\mathcal{U}_h$ as $C^\infty(\bar{\Omega})$.)

Definition 7. The form $[u, v]$ is said to be coercive on $\mathcal{U}_h$ if there exists $m > 0$ such that for each $u \in \mathcal{U}_h$:

$$\|u\|^2_{W_2^h(\Omega)} \leq m\,(\mathrm{Re}\,[u, u] + \|u\|^2_{L_2(\Omega)}) . \tag{3}$$

We denote further

$$\begin{cases} N = \{u \in \mathcal{U}_h \mid [u, v] = 0 \quad \forall v \in \mathcal{U}_h\} \\ N' = \{u \in \mathcal{U}_h \mid [v, u] = 0 \quad \forall v \in \mathcal{U}_h\} \end{cases} \tag{4}$$

and set $(\mathcal{U}_h)^h_p = \overline{\mathcal{U}_h}$ in the topology of the space $W^h_p(\Omega)$.

Proposition 8 (see [15]). Let $[u, v]$ be a coercive bilinear form on $\mathcal{U}_h$. Then there exists an isomorphism

$$\mathcal{T}: \left((\mathcal{U}_h)^h_p / N \right)^* \longrightarrow (\mathcal{U}_h)^h_q / N' , \quad \frac{1}{q} + \frac{1}{p} = 1, \quad q > 0 \tag{5}$$

defined as

$$(\mathcal{T} F = v) \overset{def}{\Longleftrightarrow} F(u) = [u, v] \quad \forall u \in \mathcal{U}_h . \tag{6}$$

Remark. The other theorems of the unique representation of linear functionals can be found in [10], [14] and others. For our purpose, the Schechter's formulation seemed us to be very convenient.

In the next part we deal with the real functions and the real spaces of functions only, if nothing else will be said.

Lemma. Let $C_1, C_2, \dots, C_h$, $0 \leq h \leq k$, be a normal set of boundary operators on $\partial\Omega$. Then the space $(\mathcal{U}_h)_p^k$, defined as above, is isomorphic with some complemented subspace of $L_p(0,1)$.

Proof. We denote by $[L_p(\Omega)]^{\varkappa}$ the Cartesian product of $\varkappa$ copies of $L_p(\Omega)$, where $\varkappa$ is the number of all multiindexes α, $|\alpha| \leq k$. Each element $(f_\alpha)_{|\alpha|\leq k}$ of $[L_p(\Omega)]^{\varkappa}$ defines uniquely $F \in ((\mathcal{U}_h)_q^k)^*$ as

$$(7) \qquad F(v) = \int_\Omega \sum_{|\alpha|\leq \varkappa} D^\alpha v \cdot f_\alpha \, dx .$$

We can extend this functional in the natural manner on the complex space. Let $\tilde{F}$ be this extension.

According to Proposition 8 we have, using the bilinear form

$$(8) \qquad [v, u] = \int_\Omega \sum_{|\alpha|\leq k} D^\alpha v \, \overline{D^\alpha u} \, dx$$

that there exists the unique $u = u_1 + i u_2$ of $(\mathcal{U}_h)_p^k$ (complex) such that

$$(9) \quad \int_\Omega \sum_{|\alpha|\leq k} D^\alpha v \cdot f_\alpha \, dx = \tilde{F}(v) = \int_\Omega \sum_{|\alpha|\leq k} D^\alpha v \, \overline{D^\alpha u} \, dx$$

for each $v \in (\mathcal{U}_h)_q^k$ (complex).

It is easy to see that $u_2 = 0$, such that we can return to $(\mathcal{U}_n)^k_p$ real. The relation (9) now gives us the continuous projection of $[L_p(\Omega)]^{\varkappa}$ onto $[(\mathcal{U}_n)^k_p]$, where

$$(10)\ [(\mathcal{U}_n)^k_p] = \{(v_\alpha)_{|\alpha|\le k} \in L_p \mid \exists u \in (\mathcal{U}_n)^k_p : v_\alpha = D^\alpha u$$

for $|\alpha| \le k \}$.

(There is a natural isomorphism $\mathcal{J}$ between $(\mathcal{U}_n)^k_p$ and $[(\mathcal{U}_n)^k_p]$.)

On the other hand, $[L_p(\Omega)]^{\varkappa}$ is isomorphic with $L_p(0,1)$. We denote this isomorphism by $\mathcal{K}$ (for the details see [7].)

Thus we can write

$$\begin{tikzcd}
(\mathcal{U}_n)^k_p \arrow[d, shift right, "\mathcal{J}"'] & \\
{[(\mathcal{U}_n)^k_p]} \arrow[u, shift right, "\mathcal{J}^{-1}"'] \arrow[d, shift right, "\mathcal{K}"'] & {[L_p(\Omega)]^{\varkappa}} \arrow[l, "P"'] \arrow[d, shift right, "\mathcal{K}"'] \\
\mathcal{K}[(\mathcal{U}_n)^k_p] \arrow[u, shift right, "\mathcal{K}^{-1}"'] & L_p(0,1) \arrow[l, "\tilde{P}"'] \arrow[u, shift right, "\mathcal{K}^{-1}"']
\end{tikzcd}$$

where $\tilde{P}$ is $\mathcal{K} \circ P \circ \mathcal{K}^{-1}$. It is obvious that $\tilde{P}$ is the continuous projection of $L_p(0,1)$ onto $\mathcal{K}[(\mathcal{U}_n)^k_p]$ which is the complemented subspace of $L_p(0,1)$. As we have

$$(\mathcal{U}_n)^k_p \xleftrightarrow{\mathcal{K}\mathcal{J}} \mathcal{K}[(\mathcal{U}_n)^k_p] ,$$

the proof of the Lemma is finished.

__Theorem__. Let $C_1, C_2, \dots, C_h$, $0 \le h \le k$, be a normal set of boundary operators on $\partial\Omega$. Then the space

$$(\mathcal{U}_h)_p^k = \overline{\{u \in C^\infty(\overline{\Omega}) \mid C_j u = 0, \ j = 1, 2, \dots, h\}}$$

where the closure is in the topology of the space $W_p^k(\Omega)$, has a Schauder basis.

__Proof__ follows immediately from Proposition 6, Proposition 7 and the Lemma.

__Remark__. Changing the boundary conditions (C_i, $i = 1, \dots, h$), we obtain (from the Theorem) the assertions about the existence of a Schauder basis for different spaces connected with boundary value problems for elliptic equations.

__Corollary 2__. The space $W_p^k(\Omega)$ has a Schauder basis. (Taking the empty set of boundary operators C_i, we obtain $(\mathcal{U}_h)_p^k = W_p^k(\Omega)$.)

__Corollary 3__. The space $\overset{\circ}{W}{}_p^k(\Omega)$ has a Schauder basis. (Taking $h = k$, $C_i = \dfrac{\partial^{i-1}}{\partial n^{i-1}}$, $i = 1, \dots, k$ and using [12, Théorème 4.13] we get $(\mathcal{U}_h)_p^k = \overset{\circ}{W}{}_p^k(\Omega)$.)

__4. Remarks and open problems__

__Problem 1__. The existence of Schauder basis in $W_1^k(\Omega)$. (Proposition 8 takes place only in the reflexive case - i.e. for $p \in (1, +\infty)$.)

__Problem 2__. The same for $W_p^k(\Omega)$ in the case that $\partial\Omega$ is not sufficiently smooth, for example lipschitzian only and for a class of closed subspaces of $W_p^k(\Omega)$.

Problem 3. The same for k which is not integer. (In [10] the theory of representation relative to the proposition 8 is proved which takes place for k being non-integer. But for such a case the representation does not give us the projection

$$[L_p(\Omega)] \xrightarrow{onto} [W_p^k(\Omega)] ,$$

so that we cannot use the results of Proposition 6 and Proposition 7.)

Problem 4. It would be useful to know how to construct any concrete basis for the space $W_p^k(\Omega)$ (or its subspace) in the case that Ω is some special domain.

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