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REMARK ON THE FREDHOLM ALTERNATIVE FOR NONLINEAR OPERATORS WITH APPLICATION TO NONLINEAR INTEGRAL EQUATIONS OF GENERALIZED HAMMERSTEIN TYPE

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§ 1. Introduction. Let B be a reflexive Banach space and T a bounded, demicontinuous mapping from B to its dual B^*. Define $T_t(u) = T(u) - tT(-u)$ and suppose that T_t satisfies for every $0 \leq t \leq 1$ the condition (S):

(1.1) If $u_n \rightharpoonup u$ (weak convergence) and $(T_t u_n - T_t u, u_n - u) \to 0$, then $u_n \to u$ (strong convergence, where $(,)$ denotes the natural pairing between B^* and B; if, for some f in B, we have also

(1.2) $T_t u - (1-t)f \neq 0$ for $\|u\| = R > 0$ and $0 \leq t \leq 1$, then there exists u in B such that $\|u\| < R$ and $Tu = f$.

Suppose T is an odd mapping and $\varkappa$-homogeneous $(T(tu) = t^{\varkappa} T(u),\ t > 0, \varkappa > 0)$ satisfying (1.1). The consequence of the above statement is the following alternative: if S is a completely continuous mapping from

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B to B^* such that

(1.3) $$\lim_{\|u\| \to \infty} \frac{\|Su\|}{\|u\|^{\varkappa}} = 0$$

and if $Tu = 0$ implies $u = 0$, then $(T+S)(B) = B^*$. Furthermore, every solution of $(T+S)u = f$ satisfies the inequality

(1.4) $$\|u\| \leq c\,(1 + \|f\|^{1/\varkappa}) .$$

Conversely, if (1.4) is true for every solution of $(T+S)u = f$, then $Tu = 0$ implies $u = 0$.

The first statement is a generalization of a result of D.G. de Figueiredo, Ch.P. Gupta [4] and the alternative is a generalization of a result of S.I. Pochožajev [10] and F. E. Browder [2], and it is another version of the author's Fredholm alternative [9]; see also M. Kučera [7] and the forthcoming papers of S. Fučík [5] and M. Kučera [8]. For T linear, we obtain a generalization of a result of M.A. Krasnoselskij [6].

Application: Let M be a measurable set in R_m with $mes\,(M) < \infty$ and ℓ an odd positive integer. For $i = 1, 2, \dots, m$, let $K_i(x, y)$ be kernels on $M \times M_\ell$, with $M_\ell = M \times M \times \dots \times M$ ℓ-times, such that

(1.5) $$\int_M \int_{M_\ell} |K_i(x, y)|^{\ell+1}\, dx\, dy < \infty .$$

Let $f_i(y, u)$ be functions defined on $M_\ell \times R_\ell$, satisfying the Caratheodory condition and the growth condition

$$(1.6)\quad |f_i(y,u)| \leq c|u|^{\ell} + d_i(y) \text{ where } d_i \in L_{1+1/\ell}(M_\ell).$$

The generalized Hammerstein's type integral equation is:

$$(1.7)\quad u^{\ell}(x) - \lambda \sum_{i=1}^{m} \int_{M_\ell} K_i(x,y) f_i(y, u(y))\, dy = g(x),$$

where $u(y) = (u(y_1), u(y_2), \dots, u(y_\ell))$ and the solution is supposed to be in $L_{\ell+1}(M)$. If the asymptotic condition: for $t \longrightarrow \infty$

$$(1.8)\quad |t^{-\ell} f_i(y, tu) - \sum_{|\alpha|=\ell} a_\alpha^i(y) u^\alpha| \leq c_i(t)|u|^{\ell} + d_i(y,t)$$

with $c_i(t) \longrightarrow 0$ and $d_i(t) \longrightarrow 0$ in $L_{1+1/\ell}(M_\ell)$ for $t \longrightarrow \infty$, where $u^\alpha = u_1^{\alpha_1} \dots u_\ell^{\alpha_\ell}$ and $a_\alpha^i \in L_\infty(M_\ell)$, then the equation (1.7) has a solution if λ is not an eigenvalue for the asymptotic homogeneous equation

$$(1.9)\quad u^{\ell}(x) - \lambda \sum_{i=1}^{m} \sum_{|\alpha|=\ell} \int_{M_\ell} K_i(x,y) a_\alpha^i(y) \,.\, u^\alpha(y)\, dy = 0 .$$

Every solution of (1.7) satisfies

$$(1.10)\qquad \|u\|_{L_{\ell+1}(M)} \leq c\,(1 + \|g\|_{L_{1+1/\ell}(M)}^{1/\ell})$$

and, conversely, if (1.10) is satisfied for every solution, then λ is not an eigenvalue of (1.9).

§ 2. Abstract Theorems

Theorem 1. Let B be a real, reflexive Banach space and T a mapping from B to B^*, bounded and demicontinuous $(u_n \longrightarrow u \Rightarrow Tu_n \rightharpoonup Tu)$. Let T satisfy the condition (1.1) and (1.2). Then there exists a solution u, $\|u\| < R$ of $Tu = f$.

Proof: Let F be a subspace of B and let ψ_F be the injection of F to B and ψ_F^* its duality mapping. Let $T_F = \psi_F^* T \psi_F$.

(i) There exists F with $dim\, F < \infty$ such that $T_{F'}(u) - t\, T_{F'}(-u) - (1-t)\, \psi_F^* f \neq 0$ for $\|u\| = R$, $u \in F' \supset F$, $0 \leq t \leq 1$, $dim\, F' < \infty$;

we prove a little more: there exists $\delta > 0$ such that $\| T_{F'}(u) - t\, T_{F'}(-u) - (1-t)\, \psi_F^* f \| \geq \delta$ for the u in question. Of course, for $w \in F^*$, $\|w\| = \sup\limits_{\substack{u \neq 0 \\ u \in F}} \dfrac{(w, u)}{\|u\|}$.

Let us prove first this statement for t fixed. Let us suppose the contrary. Then, for every F with $dim\, F < < \infty$, there exists a sequence F_n, $F \subset F_1 \subset F_2 \subset \ldots$ $\ldots$ $dim\, F_n < \infty$ and $u_n \in F_n$, $\|u_n\| = R$,

such that $\lim\limits_{n \to \infty} \| T_{F_n}(u_n) - t\, T_{F_n}(-u_n) - (1-t)\, \psi_{F_n}^* f \| \geq \delta_{t_0}$.

Suppose $u_n \rightharpoonup u$, $u \in \overline{\bigcup\limits_{n=1}^{\infty} F_n} \stackrel{df}{=} B_F$. We have $\lim\limits_{n \to \infty} (T(u_n) - T_t(u), u_n - u) = \lim\limits_{n \to \infty} (T_t(u_n) - T_t(u),$ $u_n - v_n) = \lim\limits_{n \to \infty} (T_t(u_n) - (1-t)\psi_{F_n}^* f,\ u_n - v_n) = 0$, where $v_n \longrightarrow u$, $v_n \in F_n$. Hence $u_n \longrightarrow u$ and for every $v \in B_F$: $(T_t(u) - (1-t)f, v) = 0$, hence $\| T_t(u) - (1-t)f \|_{B_F^*} = 0$. In this way, we constructed for every $F \subset B$, with $\dim F < \infty$, a separable subspace $B_F \supset F$ such that there exists $u \in B_F$, $\|u\| = R$ for which $\| T_t(u) - (1-t)f \|_{B_F^*} = 0$.

Let M_F be the set of such u corresponding to F. The set of M_F has clearly the finite intersection property. Let $\overline{M}_F$ be the closure of M_F in the weak topology. There exists $\bar{u} \in \bigcap_F \overline{M}_F$. Let F with $\dim F < \infty$ be chosen such that $w \in F$, $\bar{u} \in F$. (Compare, for example, F.E. Browder [3].) There exists $u_n \in M_F$ such that $u_n \longrightarrow \bar{u}$, $\lim_{u \to \infty} (T_t(u_n) - T_t(\bar{u}), u_n - \bar{u}) = \lim_{n \to \infty} (T_t(u_n) - (1-t)f, u_n - \bar{u}) = 0$,

hence $u_n \longrightarrow \bar{u}$. This implies $(T_t(\bar{u}) - (1-t)f, w) = 0$ and $\|\bar{u}\| = R$ which is a contradiction to (1.2). It follows that there exists, for every t_0 from the interval $\langle 0, 1 \rangle$, a set F_{t_0} with $\dim F_{t_0} < \infty$ and $\delta_{t_0} > 0$, such that if $\|u\| = R$ and $u \in F'$, $\dim F_{t_0} < \infty$, $F' \supset F_{t_0}$, then $\| T_{F'}(u) - t_0 T_{F'}(-u) - (1-t_0)\psi_{F'}^* f \| \geq \delta_{t_0}$.

Because of the boundedness of T. the same is true with F_{t_0} and $\delta_{t_0}/2$ for $|t - t_0| < \varepsilon_{t_0}$.

Hence there exists t_i , $i = 1, 2, \dots, m$, ε_{t_i} , F_{t_i} , δ_{t_i} such that $\bigcup_{i=1}^{m} \{ |t - t_i| < \varepsilon_{t_i} \} \supset \langle 0, 1 \rangle$.

If $\delta = \min \left(\frac{\delta_{t_i}}{2} \right)$ and $F = \bigcup_{i=1}^{m} F_{t_i}$, then for $F' \supset F$, $\|u\| = R$, $u \in F'$: $\| T_{F'}(u) - (1-t)\psi_{F'} - (1-t)\psi_{F'} f \| \geq \delta$, $0 \leq t \leq 1$, which is the assertion.

(ii) F chosen in (i), for $F' \supset F$, $\dim F' < \infty$, and $t = 1$, by virtue of the Borsuk-Ulam theorem, the degree $(T_{F'}(u) - T_{F'}(-u)$, $B(0, R)$, $0)$ is an odd integer. (Compare M.A. Krasnoselskij [6].) By homotopy, this is also true for $t = 0$; hence, there exists $u_{F'} \in F'$, $\|u_{F'}\| < R$, such that $T_{F'}(u_{F'}) - \psi^{*}_{F'} f = 0$. Let $M_{F'} = \{ u_{F''}, F'' \supset F' \}$.

$M_{F'}$ has the finite intersection property, hence $u \in \bigcap_{F'} \overline{M}_{F'}$, where $\overline{M}_{F'}$ is the closure in the weak topology.

Let $w \in B$, u , $w \in F'$. Then there exists $u_n \in M_{F'}$, $u_n \longrightarrow u$. $\lim (Tu_n - Tu , u_n - u) = \lim_{n \to \infty} (Tu_n , u_n - u) = \lim_{n \to \infty} (f , u_n - u) = 0$.

Hence $u_n \longrightarrow u$ and $0 = (Tu_n - f, w) \longrightarrow$
$\longrightarrow (Tu - f, w)$, q.e.d.

<u>Theorem 2</u>. Let S be a completely continuous mapping from B to B^* satisfying (1.3) and T an odd, bounded, demicontinuous and $\varkappa$ -homogeneous mapping from B to B^* . Let T satisfy the condition (1.1). Then there exists a solution of $(T + S)u =$ $= f$ and every solution satisfies the inequality (1.4) if and only if $Tu = 0 \Rightarrow u = 0$.

<u>Proof</u>: (i) Let (1.4) be true. Let us suppose there exists $u_0 \neq 0$ such that $Tu_0 = 0$. We have

$$\|u_0\| \leq c\left(\frac{1}{t} + \frac{1}{t}\, \| S(tu_0)\|^{1/\varkappa}\right) \rightarrow 0 \text{ for } t \longrightarrow \infty ,$$

which is impossible.

(ii) If $Tu = 0 \Rightarrow u = 0$, then for every solution (1.4) is true. If not, then there exists

$$u_n \in B\ ,\ u_n \longrightarrow \infty \quad \text{such that} \quad \frac{1}{n} + \frac{\|Su_n\|}{\|u_n\|} \geq$$

$$\geq \| T\left(\frac{u_n}{\|u_n\|^{\varkappa}}\right)\| \text{ . Putting } v_n = \frac{u_n}{\|u_n\|} \text{ , we can}$$

suppose $v_n \longrightarrow v$. Because $Tv_n \longrightarrow 0$, we obtain from the condition (1.1) that $v_n \longrightarrow v$ and, therefore, $Tv = 0$ and $\| v \| = 1$ which is contradictory.

(iii) $(T + S)_t$ satisfies clearly the condition (1.1).

(iv) Replace T by $T + S$ in Theorem 1. For R large enough and for every fixed f, $T + S$ satisfies the condition (1.2), q.e.d.

§ 3. Application to the integral equation (1.7)

We submit the functions $f_i(y, u)$ to the asymptotic condition (1.8). We have for $\|u\|_{L_{\ell+1}(M)} \longrightarrow \infty$:

$$\lim \|u\|^{-\ell}_{L_{\ell+1}(M)} \Big\| \int_{M_\ell} K_i(x,y)\,[f_i(y, u(y)) -$$

$$- \sum_{|\alpha|=\ell} a^i_\alpha(y) u^\alpha(y)]\, dy \Big\|_{L_{1+1/\ell}(M)} = 0 .$$

Therefore, the condition (1.3) is fulfilled for

$$S(u) \overset{df}{=} \int_{M_\ell} \sum_{i=1}^{m} K_i(x,y)\,[f_i(y,u(y)) - \sum_{|\alpha|=\ell} a^i_\alpha(y) u^\alpha(y)]\, dy .$$

The complete continuity of S follows from the well-known fact that the operator $f_i(y, u(y))$ is a continuous operator from $L_{\ell+1}(M)$ to itself; compare, for example, M.M. Vajnberg [11], and from the fact that the linear operator $\int_{M_\ell} K_i(x, y)\, w(y)\, dy$ is completely continuous from $L_{1+1/\ell}(M)$ to itself. If we put

$$T(u)(x) = u^\ell(x) + \sum_{i=1}^{m} \int_{M_\ell} K_i(x,y) \sum_{|\alpha|=\ell} a^i_\alpha(y) u^\alpha(y)\, dy ,$$

we obtain a bounded, continuous odd and ℓ-homogeneous operator from $L_{\ell+1}(M) \longrightarrow L_{1+1/\ell}(M)$.

By virtue of the complete continuity of the mapping

$\sum_{i=1}^{m} \int_{M_\ell} K_i(x,y) \sum_{|\alpha|=\ell} a_\alpha^i(y) u^\alpha(y)\,dy$, it is enough to verify the condition (S) for the duality mapping $u(x) \longrightarrow u(x)^\ell$. But

$$\int_M (u^\ell(x) - v^\ell(x))(u(x) - v(x))\,dx =$$

$$= \ell \int_M \left(\int_0^1 (v(x) + t(u(x) - v(x)))^{\ell-1} dt \right) (u(x) - v(x))^2 dx$$

$\geq c \int_M (u(x) - v(x))^{\ell+1} dx$, where we used the elementary fact that $\int_0^1 |a + \tau b|^\sigma d\tau \geq c\,|b|^\sigma$ for $\sigma \geq 0$. Hence we can use Theorem 2, and we obtain the statement from § 1.

§ 4. Hammerstein's equation

Using the result of the preceding paragraph, we obtain $L_2(M)$ theory. By virtue of the linearity of the asymptotic equation and because of the form $I + A$ of the considered operator, where I is the identity and A the completely continuous operator, we can base our consideration on the well-known fact; compare, for example, M.A. Krasnoselskij [6]:

Let B be a real Banach space and $Tu - f = (I+A)u - f$, where A is a completely continuous mapping. Let, for $\|u\| = R$: $\|Tu - f\| > 0$. Then, if the degree $(Tu - f, B(0,R), 0) \neq 0$, there exists u , $\|u\| < R$ such that $Tu = f$. It is also known (see M.A. Krasnoselskij [6]) for K a linear completely continuous operator, that the existence of $(I + K)^{-1}$ implies for $R > \|f\| \, \|(I + K)^{-1}\|$ that the degree $((I + K)u - f, B(R, 0)\, 0) = \pm 1$. Hence, by the homotopy argument, the same is true for R large enough for the operator $(I + K + S)u - f$, where $\lim_{\|u\| \to \infty} \frac{\|Su\|}{\|u\|} = 0$.

Let us consider the equation

(4.1) $$u(x) - \lambda \int_M K(x,y) f(y, u(y)) dy = g(x)$$

with $u \in L_p(M)$.

$1 \leq p \leq \infty$ and with

$$mes(M) < \infty , \int_M \int_M |K(x,y)|^{\max(p, \frac{p}{p-1})} dx dy < \infty ,$$

$1 < p < \infty$, which for $p = 1$ or $p = \infty$ would be replaced by continuity on $M \times M$ of the kernel, M assumed to be a compact set.

For $t \longrightarrow \infty$, we suppose:

(4.2) $$\left| \frac{1}{t} f(y, tu) - a(y) u \right| \leq c(t)|u| + d(y.t)$$

with $d(t) \longrightarrow 0$ in $L_p(M)$ and $c(t) \longrightarrow 0$ for $t \longrightarrow \infty$.

We have the following result (very near to the corresponding Krasnoselskij's result; compare his book [6]):

The integral equation (4.1) has a solution for every $g \in L_p(M)$ and every solution satisfies the inequality

$$(4.3) \qquad \| u \|_{L_p} \leq c\,(1 + \| g \|_{L_p}) \qquad \text{if and only if}$$

λ is not an eigenvalue of the linear equation:

$$v(x) - \lambda \int_M K(x,y)\, a(y)\, v(y)\, dy = 0 \ .$$

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