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ON GENERATION OF TORSION THEORIES

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1. Introduction. The purpose of this paper is to continue the investigation begun by Gardner [6] of torsion classes which are closed under pure subobjects. The results of the section 2 reduce the problem of classification of torsion theories for a given Abelian category to the classification of indecomposable and super-decomposable objects. In particular, the theorem 2.8 leads to a succession of typical applications in module categories which are presented in Section 3.

Let $\mathcal{C}$ be a category. A torsion theory for $\mathcal{C}$ consists of a couple $(\mathcal{M}, \mathcal{L})$ of classes of $\mathcal{C}$ which are orthogonally closed with respect to the bifunctor $Mor_{\mathcal{C}}(X, Y)$ – the set of morphisms from the object X into the object Y in the class $Ob\,\mathcal{C}$ of all objects of $\mathcal{C}$. In other words,

$\mathcal{M} = \mathcal{L}^{+} = \{M \in Ob\,\mathcal{C} \mid Mor_{\mathcal{C}}(M, L)$ consists at most of one morphism for $\forall L \in \mathcal{L}\}$,

$\mathcal{L} = \mathcal{M}^{*} = \{L \in Ob\,\mathcal{C} \mid Mor_{\mathcal{C}}(M, L)$ consists at most of one morphism for $\forall M \in \mathcal{M}\}$,

$\mathcal{M}$ is said to be the torsion class and $\mathcal{L}$ the torsion-free class. The torsion theory $(\mathcal{M}, \mathcal{L})$ is called the

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trivial torsion theory, if $\mathcal{M}$ or $\mathcal{L}$ consists of $\mathcal{O}b\,\mathcal{C}$.
If $\mathcal{C}$ is a subcomplete Abelian category, i.e. an Abelian category such that the family of subobjects $S(A)$ of any $A \in \mathcal{O}b\,\mathcal{C}$ is a set and the infinite coproducts $\coprod_{\alpha \in I} U_\alpha$ (sometimes called the direct sums and denoted by $\oplus$), and the infinite products $\prod_{\alpha \in I} \left(A/U_\alpha\right)$ (sometimes called the direct products) exist in $\mathcal{C}$ for any subset $(U_\alpha)_{\alpha \in I} \subset S(A)$ and any $A \in \mathcal{O}b\,\mathcal{C}$, then by [4],p.224 any torsion theory $(\mathcal{M}, \mathcal{L})$ for $\mathcal{C}$ possesses the properties

i) $\mathcal{M} \cap \mathcal{L} = 0$ (a zero-object),

ii) $\mathcal{M}$ is closed under quotients,

iii) $\mathcal{L}$ is closed under subobjects.

iv) For $\forall A \in \mathcal{O}b\,\mathcal{C}$ there exists a short exact sequence $0 \longrightarrow M \longrightarrow A \longrightarrow L \longrightarrow 0$ such that $M \in \mathcal{M}$ and $L \in \mathcal{L}$ and this is equivalent to the existence of the idempotent radical r (a subfunctor of identity such that $f \in Mor_{\mathcal{C}}(A, B)$ implies that $r(f)$ is the restriction of f on $r(A)$, $r \circ r = r$ and $r\left(A/r(A)\right) = 0$, for $\forall A \in \mathcal{O}b\,\mathcal{C}$) such that

$$\mathcal{M} = \{M \in \mathcal{O}b\,\mathcal{C} \mid r(M) = M\}$$

and

$$\mathcal{L} = \{L \in \mathcal{O}b\,\mathcal{C} \mid r(L) = 0\} .$$

In this case, the maximal torsion subobject of a given object A is

$$r(A) = \vee\{U_\alpha \in S(A) \mid U_\alpha \in \mathcal{M}\} = im\,\{\coprod_\alpha U_\alpha \longrightarrow A\} =$$

$= \wedge\{V_\beta \in S(A) \mid {}^A/_{V_\beta} \in \mathcal{L}\} = \ker\{A \longrightarrow \prod_\beta ({}^A/_{V_\beta})\}$,

where the image, respectively the kernel is related to the universal morphism induced by the canonical injections, respectively projections. Consider a class $\mathcal{E}$ of short exact sequences of $\mathcal{C}$, where $\mathcal{C}$ is an Abelian category such that every sequence isomorphic to a sequence in $\mathcal{E}$ is also in $\mathcal{E}$. The corresponding class of monomorphisms (epimorphisms) is written $\mathcal{E}_m$ ($\mathcal{E}_e$) . $\mathcal{E}$ is called a <u>proper class</u> (sometimes called the <u>purity</u>) if it satisfies:

i) Every split short exact sequence is in $\mathcal{E}$,

ii) If $\alpha, \beta \in \mathcal{E}_m$, then $\beta \circ \alpha \in \mathcal{E}_m$, if defined.

ii!) If $\alpha, \beta \in \mathcal{E}_e$, then $\beta \circ \alpha \in \mathcal{E}_e$, if defined.

iii) If $\beta \circ \alpha \in \mathcal{E}_m$, then $\alpha \in \mathcal{E}_m$.

iii!) If $\beta \circ \alpha \in \mathcal{E}_e$, then $\beta \in \mathcal{E}_e$.

Since $\mathcal{E}_m$ ($\mathcal{E}_e$) is closed under push-outs (pull-backs), it is equivalent to the original definition stated in [9], p. 368. If $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a proper exact sequence, we shall say that A is an $\mathcal{E}$ <u>-pure subobject</u> of B and denote it by $\underline{A \subset_{\mathcal{E}} B}$. The purity, where every subobject is pure, will be denoted by <u>*max*</u> and the class of all the split short exact sequences of $\mathcal{C}$ denote by <u>*min*</u> . Now, we are ready to introduce the term $\mathcal{E}$ <u>-essential extension</u> (see, for example [10]) for Abelian categories. For $A \subset_{\mathcal{E}} B$, we shall say that B is an $\mathcal{E}$ -essential extension of A if every $\varphi \in Mor_{\mathcal{C}}(B, X)$ such that $\varphi \circ i \in \mathcal{E}_m$, where i is the inclusion, is a monomorphism. Furthermore, if $B \in I_{\mathcal{E}}$ i.e. B is an $\mathcal{E}$ <u>-in-</u>

jective object (i.e. injective with respect to the proper class $\mathcal{E}$), we shall say that B is an $\mathcal{E}$ -injective envelope of A and denote it by $E_{\mathcal{E}}(A)$. If in the definition of $\mathcal{E}$ -essential extension we will demand the weaker condition that $\varphi \circ i$, being a monomorphism, implies that φ is a monomorphism or equivalently if a $C \in S(B)$ satisfies $C \wedge A = 0$, then $C = 0$, we will get the weak $\mathcal{E}$ -essential extension and similarly the weak $\mathcal{E}$ -injective envelope.

We now go about the task of constructing several specific torsion theories for an Abelian category $\mathcal{C}$ with respect to a given proper class $\mathcal{E}$ of short exact sequences of $\mathcal{C}$. An object P is called $\mathcal{E}$ -simple (weakly $\mathcal{E}$ -simple) if it has no $\mathcal{E}$ -pure subobjects (respectively, no $\mathcal{E}$ -pure subobjects non-isomorphic to P) except 0 and P . Let us denote the representative class of non-isomorphic $\mathcal{E}$ -simple objects (resp. weakly $\mathcal{E}$ -simple objects) by $S_{\mathcal{E}}$ (resp. $\tilde{S}_{\mathcal{E}}$). On the other hand, an object A is called $\mathcal{E}$ -thick (strongly $\mathcal{E}$ -thick) if $A \neq 0$ and there is no $\mathcal{E}$ -pure $\mathcal{E}$ -simple (weakly $\mathcal{E}$ -simple) subobject of A except zero. Let us denote the representative class of non-isomorphic $\mathcal{E}$ -thick (resp. strongly $\mathcal{E}$ -thick) objects of $\mathcal{C}$ by $c\,S_{\mathcal{E}}$ (resp. $c\,\tilde{S}_{\mathcal{E}}$). In particular, $\tilde{S}_{min}$ is the class of indecomposable objects and $c\,\tilde{S}_{min}$ will be called the class of super-decomposable objects of $\mathcal{C}$. Let $p \in S_{\mathcal{E}}$, the torsion class $T_{p,\mathcal{E}} = \{p\}^{*+}$ will be called the class of $\mathcal{E}$ - p -primary objects and similarly the torsion class $D_{p,\mathcal{E}} = \{p\}^{+}$ will be called the class of $\mathcal{E}$ - p -divisible objects. In case that $\mathcal{E} = max$, we simply shall say $\mathfrak{p}$ -primary, respectively $\mathfrak{p}$ -divisible objects. The corresponding idempotent radical for $\mathbb{T}_{\mathfrak{p},max}$ will be denoted by $\kappa_{\mathfrak{p}}$, and the corresponding torsion-free classes are $\mathbb{F}_{\mathfrak{p},\varepsilon} = \{\mathfrak{p}\}^*$ and $\mathbb{R}_{\mathfrak{p},\varepsilon} = \{\mathfrak{p}\}^{+*}$ respectively. In general, $\mathbb{D}_\varepsilon = \{S_\varepsilon\}^+$, respectively $\mathbb{R}_\varepsilon = \{S_\varepsilon\}^{+*}$ will be called the divisible, respectively the reduced objects which correspond to the torsion theory $(\mathbb{T}_\varepsilon , \mathbb{F}_\varepsilon^{\cdot})$, where $\mathbb{T}_\varepsilon = \{S_\varepsilon\}^{*+}$.

It is easy to check that the classes $\mathbb{T}_{max}$, $\mathbb{F}_{max}$, $\mathbb{D}_{max}$ and $\mathbb{R}_{max}$ coincide with the classes of ordinary torsion objects $\mathbb{T}$, ordinary torsion-free objects $\mathbb{F}$, ordinary divisible objects $\mathbb{D}$ and ordinary reduced objects $\mathbb{R}$ respectively, in the category of Abelian groups. Moreover, in the section 3 it is shown that this coincidence partially holds in the module category ${}_\Lambda Mod$, where Λ is a semi-Artinian Dedekind ring, i.e. a Dedekind ring ([2],p.134), where for any nonzero ideal I different from Λ , Λ/I has a nonzero socle $\mathfrak{s}_{max}[\Lambda/I]$. By the socle $\mathfrak{s}_\varepsilon[A]$ of an object A of the Abelian category $\mathcal{C}$, we mean the subobject of A which is defined as follows:

$$S_\varepsilon[A] = \bigvee\{M \in S(A) \mid M \approx \mathfrak{p} \ , \text{ for some } \mathfrak{p} \in S_\varepsilon\} .$$

Otherwise, we shall set the zero subobject as the socle. Similarly, we can define the $\mathfrak{p}$ -socle

$$S_{\varepsilon,\mathfrak{p}}[A] = \bigvee\{M \in S(A) \mid M \approx \mathfrak{p}\} \text{, for a } \mathfrak{p} \in S_\varepsilon \ .$$

Whenever we replace S_ε by $\tilde{S}_\varepsilon$, we will attach to the corresponding term the wavy line, for example $\tilde{\mathbb{T}}_\varepsilon = \{\tilde{S}_\varepsilon\}^{*+}$. We shall say that an idempotent radical κ is an ε -torsion radical if $r(A) \subset_{\mathcal{E}} A$, for $\forall A \in \mathcal{O}b\, \mathcal{C}$ and $A \subset_{\mathcal{E}} B$ implies that $r(A) = A \wedge r(B)$. Obviously, it implies that the corresponding torsion class $\mathcal{M}$ is closed under $\mathcal{E}$ - pure subobjects.

We will frequently use the following notation:

K_{Λ} - the field of quotients of an entire Λ ,

$\mathbb{Z}$ - the entire of integers,

$\mathbb{Q} = K_{\mathbb{Z}}$ - the full rational group,

$\mathbb{P}$ - the set of all prime integers,

$\mathbb{Q}(I) = \{\frac{m}{n} \in \mathbb{Q} \mid n$ is prime to every $p \in \mathbb{P} \setminus I\}$ for $I \subset \mathbb{P}$,

$\mathbb{Z}(p^{k})$, $1 \leq k < \infty$ - the cyclic p -primary Abelian group of order p^{k} , for $p \in \mathbb{P}$.

$\mathbb{Z}(p^{\infty})$ - p -Prüfer Abelian group, for $p \in \mathbb{P}$,

A_{p} - the ring of p -adic integers, for $p \in \mathbb{P}$.

A product (resp. coproduct) of copies of an object A will be denoted by $\underline{(A)^{I}}$ (resp. $\underline{(A)^{(I)}}$), where I is the index-set. Whenever in this paper Ext , respectively Tor , will appear, it will have the usual meaning it has in the homological algebra.

2. On generation of torsion theories

Proposition 2.1. Let $\mathcal{C}$ be a subcomplete Abelian category which possesses a generator $\mathcal{U}$ and $(\mathcal{M}, \mathcal{L})$ a non-trivial torsion theory for $\mathcal{C}$. Then $\mathcal{U} \notin \mathcal{M}$. Moreover, if the generators are closed under taking nonzero subobjects, then $\mathcal{U} \in \mathcal{L}$.

<u>Proof</u>. Let $L \in \mathcal{L}$. By the definition of generator [1],p.113 there exists a morphism $f \in Mor_{\mathcal{E}}(\mathcal{U}, L)$ which cannot be factorized through the zero subobject. Moreover, if the generators are closed under taking nonzero subobjects, then the largest torsion subobject $\kappa(\mathcal{U}) \subset \mathcal{U}$ has to be necessarily zero, i.e. $\mathcal{U} \in \mathcal{L}$, q.e.d.

<u>Corollary 2,2</u>. Any nontrivial torsion-free class of Abelian groups contains all the free Abelian groups and consequently, the class of Abelian groups which have no free direct summand forms the largest nontrivial torsion class in Abelian groups.

The proof of the following proposition is straightforward and hence omitted.

<u>Proposition 2.3</u>. Let $\mathcal{E}_1 \subset \mathcal{E}_2$ be two proper classes of short exact sequences of the Abelian category $\mathcal{C}$. Then: i) There exist $S_{\mathcal{E}_1}$ and $S_{\mathcal{E}_2}$ such that $S_{\mathcal{E}_2} \subset S_{\mathcal{E}_1}$, ii) $\mathbb{T}_{\mathcal{E}_1} \supset \mathbb{T}_{\mathcal{E}_2}$ and $\mathbb{R}_{\mathcal{E}_1} \supset \mathbb{R}_{\mathcal{E}_2}$, iii) $\mathbb{F}_{\mathcal{E}_1} \subset \mathbb{F}_{\mathcal{E}_2}$ and $\mathbb{D}_{\mathcal{E}_1} \subset \mathbb{D}_{\mathcal{E}_2}$, iv) $S_{\mathcal{E}_1}[A] \supset$ $\supset S_{\mathcal{E}_2}[A]$, for $\forall A \in Ob\,\mathcal{C}$. Moreover, if $\mathcal{E}$ is an arbitrary proper class of short exact sequences, then for any $S_{\mathcal{E}}$ and $c\,S_{\mathcal{E}}$ there exist $\widetilde{S}_{\mathcal{E}}$ and $c\,\widetilde{S}_{\mathcal{E}}$ such that $S_{\mathcal{E}} \subset \widetilde{S}_{\mathcal{E}}$ and $c\,\widetilde{S}_{\mathcal{E}} \subset c\,S$.

<u>Proposition 2.4</u>. Let $\mathcal{E}$ be a proper class of short exact sequences of the subcomplete Abelian category $\mathcal{C}$ which has the $\mathcal{E}$ <u>-injective envelopes</u> and let $(\mathcal{M}, \mathcal{L})$ be a nontrivial torsion theory for $\mathcal{C}$. If $\mathcal{L}$ is closed under ta-

king $\mathcal{E}$ -injective envelopes, then $\mathcal{M}$ is closed under $\mathcal{E}$ -pure subobjects. Conversely, if $\mathcal{M}$ is closed under $\mathcal{E}$ -pure subobjects and the corresponding idempotent radical is $\mathcal{E}$ -torsion, then $\mathcal{L}$ is closed under taking $\mathcal{E}$ -injective envelopes.

<u>Proof</u>. Let $\mathcal{L}$ be closed under taking $\mathcal{E}$ -injective envelopes and let $L \in \mathcal{L}$ be arbitrary. If $M \in \mathcal{M}$ and $N \subset_{\mathcal{E}} M$, we have the exact sequence

$$Mor_{\mathcal{C}}(M, E_{\mathcal{E}}(L)) = 0 \longrightarrow Mor_{\mathcal{C}}(N, E_{\mathcal{E}}(L)) \longrightarrow$$
$$\longrightarrow \mathcal{E} - Ext^1_{\mathcal{C}}(M/N, E_{\mathcal{E}}(L)) = 0$$

which implies $Mor_{\mathcal{C}}(N, E_{\mathcal{E}}(L)) = 0$, and since the functor $Mor_{\mathcal{C}}(N, \cdot)$ is left-exact, we have $Mor_{\mathcal{C}}(N, L) = 0$.

Now, let the converse-conditions be satisfied and let $L \in \mathcal{L}$ be arbitrary. Then $\kappa(E_{\mathcal{E}}(L)) \wedge L = \kappa(L) \subset_{\mathcal{E}} L$ and since $L \subset_{\mathcal{E}} E_{\mathcal{E}}(L)$, we have $\kappa(E_{\mathcal{E}}(L)) \wedge L \subset_{\mathcal{E}}$ $\subset_{\mathcal{E}} \kappa(E_{\mathcal{E}}(L))$, i.e. $\kappa(E_{\mathcal{E}}(L)) \wedge L \in \mathcal{M} \cap \mathcal{L} = 0$ which implies $\kappa(E_{\mathcal{E}}(L)) = 0$, q.e.d.

<u>Corollary 2.5</u>. The torsion classes $T_{\pi,\mathcal{E}}$ and $T_{\mathcal{E}}$ are closed under $\mathcal{E}$ -pure subobjects provided that $\mathcal{C}$ satisfies the conditions given in the prop.2.4 and $\mathcal{C}$ has weak- $\mathcal{E}$ -injective envelopes.

<u>Proof</u>. By the prop. 2.4 it is sufficient to show that $F_{\mathcal{E}}$ and $F_{\pi,\mathcal{E}}$ are closed under taking $\mathcal{E}$ -injective envelopes. Precisely, according to the proof of the proposition 2.4, it is sufficient to show that $F_{\pi,\mathcal{E}}$ and $F_{\mathcal{E}}$

are closed under taking weak $\mathcal{E}$ -injective envelopes.

Let $F \in \mathbf{F}_{p,\mathcal{E}}$ and $\overline{E}_{\mathcal{E}}(F)$ be its weak $\mathcal{E}$ -injective envelope. If $f \in Mor_{\mathcal{C}}(\pi, \overline{E}_{\mathcal{E}}(F))$, then $im f \wedge F$ is necessarily zero since otherwise $p \approx im f \wedge F \in \mathbf{F}_{p,\mathcal{E}}$ yields the contradiction. Hence $im f = 0$. The case of $\mathbf{F}_{\mathcal{E}}$ can be proved in a similar way, q.e.d.

Proposition 2.6. Let $\mathcal{E}$ be a proper class of short exact sequences of the Abelian category $\mathcal{C}$. Then:

i) If $\widetilde{\mathbf{D}}_{\mathcal{E}}$ is closed under taking $\mathcal{E}$ -pure subobjects and $\mathcal{M}$ is another such a torsion class, then there exists a representative class Θ of non-isomorphic objects of $\widetilde{\mathbf{D}}_{\mathcal{E}} \cap \mathcal{M}$ such that $\Theta \subset c\widetilde{S}_{\mathcal{E}}$.

ii) There exists a representative class $\mathcal{D}$ of non-isomorphic objects of $\widetilde{\mathbf{D}}_{min}$ such that $\mathcal{D} \subset c\widetilde{S}_{min}$.

Proof. i) Let $M \in \widetilde{\mathbf{D}}_{\mathcal{E}} \cap \mathcal{M}$ and $M \neq 0$, then by the definition of $\widetilde{\mathbf{D}}_{\mathcal{E}}$, M is not weakly $\mathcal{E}$ -simple and since $\widetilde{\mathbf{D}}_{\mathcal{E}}$ is closed under $\mathcal{E}$ -pure subobjects, M is strongly $\mathcal{E}$ -thick.

ii) It is an immediate consequence of i).

Proposition 2.7. Let $\mathcal{C}$ be an Abelian category. Then:

i) If $A \in \mathbf{T}_{max}$ (resp. $\mathbf{T}_{p,max}$), then A is an essential extension of its socle (resp. its p -socle).

ii) If p, $p' \in S_{max}$ with $p \neq p'$, then $\mathbf{T}_{p,max} \cap \mathbf{T}_{p',max} = \{0\}$.

iii) If $A \in \mathbf{T}_{p,max}$, then A is p'-divisible for

$p' \neq p$ in S_{max} .

iv) If $A \in T_{p,max}$ and p -divisible, then $A \in \mathbb{D}_{max}$.

Furthermore, if $\mathscr{C}$ is subcomplete with injective envelopes, then the primary decomposition of torsion objects from T_{max} holds iff for each $p \in S_{max}$, the functor $A \longmapsto \kappa_p(A)$ is exact on the full subcategory $\mathbb{T}_{max}$.

Proof. [4], pp.230,234.

Theorem 2.8. Let $\mathcal{E}$ be a proper class of short exact sequences of an Abelian category $\mathscr{C}$ and let $\mathcal{M}$ be a torsion class in $\mathscr{C}$ which is closed under $\mathcal{E}$ -pure subobjects. Then

$$\mathcal{M} = \{(\mathcal{M} \cap \tilde{\mathfrak{S}}_{\mathcal{E}}) \cup (\mathcal{M} \cap c\tilde{\mathfrak{S}}_{\mathcal{E}})\}^{*+} .$$

Proof. Let $\mathcal{M}' = \{(\mathcal{M} \cap \tilde{\mathfrak{S}}_{\mathcal{E}}) \cup (\mathcal{M} \cap c\tilde{\mathfrak{S}}_{\mathcal{E}})\}^{*+}$.

Let us denote the torsion-free class corresponding to $\mathcal{M}'$ by $\mathcal{L}'$ and let $L \in \mathcal{L}'$ be arbitrary. It is sufficient to prove that $Mor_{\mathscr{C}}(M, L) = 0$, for $\forall M \in \mathcal{M}$. Let $f \in Mor_{\mathscr{C}}(M, L)$, then $im f \in \mathcal{M} \cap \mathcal{L}'$ and since $\mathcal{M}$ is closed under $\mathcal{E}$ -pure subobjects and $\mathcal{M}' \cap \mathcal{L}' = \{0\}$, $im f$ has no non-zero $\mathcal{E}$ -pure subobject either from $\tilde{\mathfrak{S}}_{\mathcal{E}}$ or $c\tilde{\mathfrak{S}}_{\mathcal{E}}$, hence $im f = 0$, q.e.d.

Corollary 2.9. Any torsion theory $(\mathcal{M}, \mathcal{L})$ for an arbitrary Abelian category $\mathscr{C}$ yields the equality

$$\mathcal{M} = \{(\mathcal{M} \cap \tilde{\mathfrak{S}}_{min}) \cup (\mathcal{M} \cap c\tilde{\mathfrak{S}}_{min})\}^{*+} .$$

3. An application

The proof of the following proposition is straightforward and hence omitted.

Proposition 3.1. Let Λ be a unitary ring. Then the following assertions hold in the category ${}_{\Lambda}\mathcal{M}od$:

i) $c\tilde{\mathcal{S}}_{max} \neq \emptyset$ iff there exists a left-ideal $\mathcal{J} \subset \Lambda$ such that $\Lambda/\mathcal{J} \in c\tilde{\mathcal{S}}_{max}$,

ii) $\mathcal{S}_{max}$ consists of the quotients $\Lambda/\mathcal{J}$, where $\mathcal{J}$ are the maximal left ideals.

iii) $\tilde{\mathcal{S}}_{max}$ can be chosen such that it consists of $\mathcal{S}_{max}$ and representatives of quotients $\Lambda/\mathcal{J}$, where $\mathcal{J}$ are such left ideals that every non-zero submodule of $\Lambda/\mathcal{J}$ is isomorphic to $\Lambda/\mathcal{J}$.

Corollary 3.2. Let Λ be a semi-Artinian ring. Then the following assertions hold in the category ${}_{\Lambda}\mathcal{M}od$:

i) $c\tilde{\mathcal{S}}_{max} \neq \emptyset$ iff $\Lambda \in c\tilde{\mathcal{S}}_{max}$,

ii) $\mathcal{S}_{max} \subset \tilde{\mathcal{S}}_{max} \subset (\mathcal{S}_{max} \cup \{\Lambda\}) \cup \{0\}$.

Proof. The assertions i) and ii) follow directly from the definition of the semi-Artinian ring and the proposition 3.1.

Let Λ be a commutative entire (i.e. an integral domain) and let $W \subset \Lambda$ be a subset. For any $M \in \mathcal{O}b\,{}_{\Lambda}\mathcal{M}od$, we can introduce the following binary relation on the lattice of submodules of M :

$\mathcal{U}_1 \sim \mathcal{U}_2 \Longleftrightarrow \mathcal{U}_1 \subset \mathcal{U}_2$ and $\alpha \cdot \mathcal{U}_1 = \mathcal{U}_1 \cap \alpha \cdot \mathcal{U}_2$,

for each $\alpha \in W$ provided that $\mathcal{U}_1 , \mathcal{U}_2 \in S(M)$.

It is an essential routine to show that this relation defines a purity on ${}_{\Lambda}Mod$ by the definition:

$\mathcal{U}$ is the pure submodule of M iff $\mathcal{U} \sim M$.

The proofs are similar to those in [5],p.78, and hence omitted. We shall say that $\mathcal{U}$ is W -pure in M and denote it by $\mathcal{U} \subset_W M$. Consequently, $\mathcal{U}$ is pure in M if $W = \Lambda$.

Proposition 3.3. Let Λ be a non-simple commutative semi-Artinian Dedekind ring such that ${}_{\Lambda}Mod$ holds the primary decomposition of torsion modules from $\mathbb{T}_{max}$. Then

$$\tilde{S}_{\mathcal{E}_W} \subset S(K_{\Lambda}) \cup \left(\bigcup_{p \in S_{max}} [(\mathbb{T}_{p,max} \cap \mathbb{T}_{max}) \cup (\mathbb{T}_{p,max} \cap \mathbf{R})] \right),$$

where $\mathcal{E}_W$ denotes the proper class of short exact sequences in ${}_{\Lambda}Mod$ corresponding to the W -purity, for some $W \subset \Lambda$.

The proof is based on the following useful lemma.

Lemma 3.4. Let Λ be a commutative non-simple entire. Then $\mathbb{T}_{max} = \mathbf{T}$ and $\mathbf{F}_{max} = \mathbf{F}$ in ${}_{\Lambda}Mod$ iff Λ is a semi-Artinian ring.

Proof. Since $(\mathbf{T}, \mathbf{F})$ is a torsion theory for ${}_{\Lambda}Mod$, it is sufficient to prove that $\mathbf{F}_{max} = \mathbf{F}$. Let $\underline{m}$ be a maximal ideal of Λ and let $F \in \mathbf{F}$ be arbitrary. If $f \in Hom_{\Lambda}(\Lambda/\underline{m}, F)$, then the annihilator of

$im f$, $ann\,(im f) \supset \underline{m} \neq 0$ and since $im f \in \mathbb{F}$, we have $im f = 0$ i.e. $\mathbb{F} \subset \mathbb{F}_{max}$ regardless if Λ is semi-Artinian or not. Hence we can restrict the proof as follows. Let $\mathbb{F}_{max} \subset \mathbb{F}$ and let I be a nonzero ideal different from Λ. Then $\Lambda/_I \in \mathbb{T} \subset \mathbb{T}_{max}$, so $\Lambda/_I$ possesses a non-zero socle (by the proposition 2.7 i)), i.e. Λ is semi-Artinian since I was arbitrary. Conversely, let Λ be semi-Artinian and let $F' \in \mathbb{F}_{max}$. Suppose that $F' \notin \mathbb{F}$, then there exists an element $x \in F'$ with $ann\,(x) \neq 0$ and $\Lambda \cdot x \approx \Lambda/_{ann\,(x)} \in \mathbb{F}_{max}$.

By the hypothesis, there is a maximal ideal $\underline{m}$ such that $\Lambda/_{\underline{m}} \subset \Lambda/_{ann\,(x)}$ i.e. $\Lambda/_{\underline{m}} \in \mathbb{F}_{max}$ and it leads to the contradiction, q.e.d.

<u>Proof of 3.3</u>. Let $M \in \tilde{S}_{\varepsilon_W}$. Since its maximal ordinary torsion submodule M_t is pure, it is consequently W-pure in M and hence M is not a mixed module. Now, suppose that $M \in \mathbb{T} \cap \tilde{S}_{\varepsilon_W}$. Since the primary decomposition for $\mathbb{T}_{max}$ holds in ${}_\Lambda Mod$ and by the lemma 3.4 $M \in \mathbb{T}_{max}$, we have the result $M \in \mathbb{T}_{p,max}$, for some $p \in S_{max}$. Let us denote the maximal ordinary divisible submodule of M by $\mathbb{D}$. Since Λ is the Dedekind ring, $\mathbb{D}$ is an injective submodule ([2],p.134) and consequently $M = \mathbb{D} \oplus \mathbb{R}$, where $\mathbb{R} \in \mathbb{R}$. In other words,

$$M \in \Big(\bigcup_{p \in S_{max}} [(T_{p,max} \cap T_{max}) \cup (T_{p,max} \cap R)] \Big) .$$

To finish the proof, let us assume that $M \in F \cap \tilde{S}_{\varepsilon_W}$ and $x \in M$ be a nonzero element. By the essentially same routine as in [5],p.78 we can show that $\Lambda \cdot x$ can be imbedded in a W -pure submodule of the rank 1 and since every Λ -module from F of the rank 1 can be imbedded in K_Λ , the proof is finished, q.e.d.

Corollary 3.5. Let $\Lambda = Z$ and $W \subset P$. Then

$$c\tilde{S}_{\varepsilon_{\overline{W}}} = \emptyset \quad \text{and}$$

$$\tilde{S}_{\varepsilon_{\overline{W}}} = \{ Z(p) \mid p \in P \} \cup \{ Z(p^k) \mid p \in W, 1 < k \leq \infty \} \cup$$
$$\cup \{ Q(I) \mid I \subset W \} \cup \{ 0 \}$$

where $\overline{W} = \{ n \in Z \mid n = \prod_{i \in K} p_i^{\alpha_i} \, , \text{ for } p_i \in W$ and K -finite $\}$.

Proof. Obviously, S_{max} and $\{ Z(p^k) \mid p \in W ,$ $1 < k < \infty \}$ are both contained in $\tilde{S}_{\varepsilon_{\overline{W}}}$. If $p \in W$, then any W -pure subgroup of $Z(p^\infty)$ is divisible and since $Z(p^\infty)$ is indecomposable, it is necessarily a member of $\tilde{S}_{\varepsilon_{\overline{W}}}$. Similarly, any W -pure subgroup of $Q(I)$, $I \subset W$, is p -divisible for any $p \in I$, so it is of the same type as $Q(I)$ and hence isomorphic to $Q(I)$ ([5],p.149), i.e. $Q(I) \in \tilde{S}_{\varepsilon_{\overline{W}}}$.

Conversely, let $M \in \tilde{S}_{\varepsilon_{\overline{W}}} \setminus S_{max}$. Since every ordinary torsion, reduced group has a finite cyclic direct summand ([8],p.21), we can use the proposition 3.3 with the result

$M \in \{Q(I) \mid I \subset \mathbb{P}\} \cup \{Z(p^k) \mid p \in \mathbb{P},\ 1 < k \leq \infty\} \cup \{0\}$.

If $I \not\subset W$, then obviously $Q(I \cap W) \subset_{\overline{W}} Q(I)$, hence $Q(I) \notin \tilde{S}_{\varepsilon_{\overline{W}}}$ and similarly if $p \notin W$, then $Z(p) \subset_{\overline{W}} Z(p^k)$, for $1 < k \leq \infty$.

Now, since every infinite cyclic subgroup can be imbedded in a pure subgroup of the rank 1 ([5],p.78) and the maximal ordinary torsion subgroup is pure, too, we have the result $c\tilde{S}_{\varepsilon_{\overline{W}}} = \emptyset$, that immediately follows from the first part of the proof, q.e.d.

Corollary 3.6. Let $\Lambda = \mathbb{Z}$ and $W \subset \mathbb{P}$. Then any torsion class $\mathcal{M} \neq \{0\}$ which is closed under taking $\overline{W}$ - pure subgroups can be described as follows.

i) $\mathcal{M} = \{(\bigcup_{p \in J_1 \subset \mathbb{P}} Z(p)) \cup (\bigcup_{p \in J_2 \subset W \setminus J_1} Z(p^\infty))\}^{*+}$,

or

ii) $\mathcal{M} = \{(\bigcup_{p \in J_3 \subset E} Z(p)) \cup Q(E)\}^{*+}$,

where $E \subset W$.

A part of the proof is the following useful lemma.

Lemma 3.7. Let $(\mathcal{M}, \mathcal{L})$ be a non-trivial torsion theory for Abelian groups such that $\mathcal{M} \cap \mathbb{F} \neq \{0\}$. Then $\mathbb{D} \subset \mathcal{M}$ and $\mathcal{L} \subset \mathbb{R}$.

Proof. Let $M \in \mathcal{M} \cap \mathbb{F}$, $M \neq 0$. Then by [2], p.119, $Hom(Q \otimes M, L) \approx Hom(Q, Hom(M, L)) = 0$, for $\forall L \in \mathcal{L}$. Since the functor $Q \otimes (\cdot)$ is exact and Q is injective we have the inclusion

$$Hom(\mathbb{Q},L) \approx Hom(\mathbb{Q}\otimes\mathbb{Z},L) \subset Hom(\mathbb{Q}\otimes M,L) = 0$$

and hence $L \in \mathbb{R}$, q.e.d.

<u>Proof of 3.6</u>. By the lemma 3.7, we can divide our investigation into the two following cases:

i) $\mathcal{M} \cap \mathbb{F} = \{0\}$.

By [3],p.31 $Z(p^k) \in \mathcal{M}$, for some $1 \leq k < \infty$ implies that $\mathcal{M}$ contains all the p -groups and hence the equality

$$\mathcal{M} = \{(\bigcup_{p \in J_1 \in \mathbb{P}} Z(p)) \cup (\bigcup_{p \in J_2 \subset W \setminus J_1} Z(p^\infty))\}^{*+}$$

follows directly from the theorem 2.8 and the corollary 3.5.

ii) $\mathcal{M} \cap \mathbb{F} \neq \{0\}$.

By the lemma 3.7, $\mathcal{M}$ contains all the divisible groups and hence with regard to the same arguments as in i), we can restrict our investigation to the case

$$\mathcal{M} = \{(\bigcup_{p \in J_3 \subset \mathbb{P}} Z(p)) \cup (\bigcup_{I \in \mathcal{P}'} \mathbb{Q}(I))\}^{*+} ,$$

where $\mathcal{P}'$ is a subset of the power set $\mathcal{P}(W)$. Since $I_1 \subset \subset I_2$ and $\mathbb{Q}(I_1) \in \mathcal{M}$ implies $\mathbb{Q}(I_2) \in \mathcal{M}$ and since by the essentially same argument as in [6],p.112, $\mathbb{Q}(I_1)$ and $\mathbb{Q}(I_2) \in \mathcal{M}$ imply $\mathbb{Q}(I_1 \cap I_2) \in \mathcal{M}$, we can rewrite the original equality as follows:

$$\mathcal{M} = \{(\bigcup_{p \in J_3 \subset \mathbb{P}} Z(p)) \cup \mathbb{Q}(\bigcap_{I \in \mathcal{P}'} I)\}^{*+} .$$

If we set $E = \bigcap_{I \in \mathcal{P}'} I$, then $\mathbb{Q}(E)/p.\mathbb{Q}(E)$,for $p \notin E$ are nonzero groups of bounded order and hence it implies that $\mathcal{M}$ contains all the p -groups, for $p \notin E$. In other words,

$\mathcal{M} = \{ (\bigcup_{p \in J_{\mathcal{E}} \subset E} Z(p)) \cup Q(E) \}^{*+}$, where $E \subset W$, q.e.d.

Let us introduce several important notations for Abelian groups. If G is an Abelian group, its Ext <u>-completion</u> is the Abelian group $\mathrm{Ext}(Q/Z, G)$ which appears to be a direct summand of a direct product of p-adic integers provided that $G \in F \cap R$ ([7], p.369). An Abelian group G is called the <u>cotorsion group</u> if $\mathrm{Ext}(Q, G) = 0$, moreover if $G \in R$, then $G \approx \mathrm{Ext}(Q/Z, G)$.

We shall say that a reduced, ordinary torsion-free and cotorsion group G is <u>of the type</u> $J \subset P$, if G is a direct summand of a direct product $\prod_{p \in J} A_p$ of p-adic integers. Of course, the type J is not uniquely determined. The following two propositions appear to be useful tools for an investigation of torsion theories for Abelian groups.

<u>Proposition 3.8</u>. Let $(\mathcal{M}, \mathcal{L})$ be a torsion theory for Abelian groups such that $\mathcal{M} \cap F \neq \{0\}$. Then $\mathcal{M}$ is closed under the Ext -completion.

<u>Proof</u>. Since the Ext -completion of divisible groups is zero, it is sufficient to show that $M \in \mathcal{M} \cap R$ implies $\mathrm{Ext}(Q/Z, M) \in \mathcal{M}$. Let $M \in \mathcal{M} \cap R$. We have the exact sequence

$$\mathrm{Hom}(Q, M) = 0 \longrightarrow \mathrm{Hom}(Z, M) \approx M \longrightarrow \mathrm{Ext}(Q/Z, M) \longrightarrow$$
$$\longrightarrow \mathrm{Ext}(Q, M) \longrightarrow 0 = \mathrm{Ext}(Z, M)$$

that yields the equivalence

$$\mathrm{Hom}(\mathrm{Ext}(Q/Z, M), L) \approx \mathrm{Hom}(\mathrm{Ext}(Q, M), L) \quad ,$$

for $\forall L \in \mathcal{L}$.

By [5],p.245, $Ext(\mathbb{Q}, M)$ is divisible and according to Lemma 3.7, $Hom(Ext(\mathbb{Q}/\mathbb{Z}, M), L) = 0$, for $\forall L \in \mathcal{L}$, q.e.d.

Proposition 3.9. Let $(\mathcal{M}, \mathcal{L})$ be a torsion theory for Abelian groups such that $\{A_p \mid p \in J \subset \mathbb{P}\} \subset \mathcal{M}$. Then $\mathcal{M}$ contains every reduced, ordinary torsion-free and cotorsion group of the type J.

Proof. If $J = \emptyset$ there is nothing to prove, so we will assume that $J \neq \emptyset$. First, we will prove that an arbitrary direct product $(A_p)^I$ of copies of A_p, for $p \in J$ belongs to $\mathcal{M}$. Since $(A_p)^I$ is equipped with the ring A_p and the height of every $x \in (A_p)^I$ is finite, we have for each $x \in (A_p)^I$ the A_p-module $A_p \cdot x$ which is isomorphic to A_p ([5],p.155). Hence we have the natural epimorphism

$$\varphi : \coprod_{x_\alpha \in (A_p)^I} A_p \cdot x_\alpha \longrightarrow (A_p)^I$$

$$(a_\alpha x_\alpha)_{\alpha \in K\text{-finite}} \longmapsto \sum_{\alpha \in K} a_\alpha x_\alpha$$

which finishes the first part of the proof. Hence any direct product of p-adic integers A_p, $p \in J$ can be written as the direct product $\prod_{p \in J} \mathcal{R}_p$, where $\mathcal{R}_p = (A_p)^{J_p}$ and this induced direct product is without repetitions. We have just shown that such an $\mathcal{R}_p \in \mathcal{M}$, so it is sufficient to prove that $\prod_{p \in J} \mathcal{R}_p \in \mathcal{M}$ provided that J is an infinite subset of $\mathbb{P}$.

Since $\coprod_{p \in J} \mathcal{R}_p \in \mathcal{M}$, we have the equivalence

$$Hom\left(\prod_{p\in J}\mathcal{R}_p/\coprod_{p\in J}\mathcal{R}_p, L\right)\approx Hom\left(\prod_{p\in J}\mathcal{R}_p, L\right),$$

for $\forall L\in\mathcal{L}$.

It is easy to show that J being infinite implies that $\prod_{p\in J}\mathcal{R}_p/\coprod_{p\in J}\mathcal{R}_p$ is divisible and since by the lemma 3.7 $\mathcal{L}\subset\mathbb{R}$, the whole proof is finished, q.e.d.

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