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CONCERNING ALMOST DIVISIBLE TORSION FREE ABELIAN GROUPS

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A torsion free group G (all groups here are supposed to be abelian) will be called almost divisible if the set of all positive primes p with $pG \neq G$ is finite. In this note we shall give some conditions that are necessary and sufficient for an almost divisible group G to be completely decomposable. In the paper [2] of D.K. Harrison (see Proposition 5.2) such necessary and sufficient conditions are formulated for the groups of finite rank. But it was shown later (see [3]) that these conditions are not sufficient in general. However, the remark following Theorem 2 shows that the Harrison's conditions are sufficient whenever the corresponding type set is linearly ordered.

If G is a torsion free group, then $\mathcal{T}(G)$ will denote the type set of all non zero elements in G ; G is said to be homogeneous of the type u if $\mathcal{T}(G)$ consists of one element u only. For a type u and a prime p the relation $u(p) = \infty$ means that in any height belonging to u the p-height is ∞ ; the symbols $G(u)$, $G^*(u)$ and $G^{**}(u)$ represent the subgroups of G defined in

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[1,§42]. The rank of a group G is denoted by $r(G)$ and $r_p(G)$ stands for its p-rank (see [5]). But in this note we shall use the relation $r_p(G) = 0$ only; this last relation says that for any finite set $M \subseteq G$ the p-primary component of the torsion group $\{M\}_*/\{M\}$ is reduced ($\{M\}_*$ denotes here the least pure subgroup of G containing M).

First of all we shall prove the following helpful assertion.

<u>Lemma.</u> Let G be an almost divisible torsion free group and let, for a type $\mathfrak{u} \in \mathcal{T}(G)$, the following conditions be fulfilled:

(a) The group $G(\mathfrak{u}) / G^*(\mathfrak{u})$ is torsion free and belongs to some Baer's class Γ_α ;

(b) for any prime p the inequality $\mathfrak{u}(p) \neq \infty$ implies

$$r_p(G(\mathfrak{u})/G^*(\mathfrak{u})) = 0 \quad .$$

Then the group $G^*(\mathfrak{u})$ is a direct summand of $G(\mathfrak{u})$, $G(\mathfrak{u}) = G_{\mathfrak{u}} \dotplus G^*(\mathfrak{u})$, where $G_{\mathfrak{u}}$ is completely decomposable and homogeneous of the type $\mathfrak{u}$, or $G_{\mathfrak{u}} = 0$.

<u>Proof</u>. If $G^*(\mathfrak{u}) = G(\mathfrak{u})$, then $G_{\mathfrak{u}} = 0$, therefore we may suppose that $G^*(\mathfrak{u}) \neq G(\mathfrak{u})$. The group $G(\mathfrak{u})$ as a pure subgroup of G is likewise almost divisible and so is the factor group $\overline{G} = G(\mathfrak{u}) / G^*(\mathfrak{u})$ as well. In view of (a), the group $\overline{G}$ is torsion free and the type of any of its non zero elements is $\geq \mathfrak{u}$. Thus, if p is a prime with $\mathfrak{u}(p) = \infty$, then $p\overline{G} = \overline{G}$. But if $\mathfrak{u}(p) \neq \infty$, then by (b) $r_p(\overline{G}) = 0$.
In the last case, each pure rank one subgroup of $\overline{G}$ is

of zero p-rank (see [4,Corollary 2]), therefore each non zero element of $\overline{G}$ has a finite p-height in $\overline{G}$ (see [6,Lemma 6.1]). Now we deduce from the finiteness of the set of all primes p with $\mathfrak{u}(p) \neq \infty$ that $\overline{G}$ is homogeneous of the type $\mathfrak{u}$. Thus the inequality $p\overline{G} \neq \overline{G}$ implies $\mathfrak{u}(p) \neq \infty$ and therefore $\kappa_p(\overline{G}) = 0$. By (a), $\overline{G}$ belongs to some Baer's class Γ_α and in view of [4,Corollary 4] $\overline{G}$ is completely decomposable. Evidently, $\mathfrak{u}$ is the type of any element $g \in G(\mathfrak{u}) \dot{-} G^*(\mathfrak{u})$, hence, according to the Baer's lemma [1,the note following Theorem 46.5] $G^*(\mathfrak{u})$ is a direct summand in $G(\mathfrak{u})$. Thus we have $G(\mathfrak{u}) = G_{\mathfrak{u}} \dot{+} G^*(\mathfrak{u})$, $G_{\mathfrak{u}} \cong G(\mathfrak{u})/G^*(\mathfrak{u}) = \overline{G}$, therefore $G_{\mathfrak{u}}$ is completely decomposable and homogeneous of the type $\mathfrak{u}$.

Now we are in a position to prove a theorem concerning almost divisible groups with the linearly ordered type set (in natural order of the types).

<u>Theorem 1</u>. Let G be an almost divisible torsion free group with the linearly ordered type set $\mathcal{T}(G)$. Then G is completely decomposable if and only if for any $\mathfrak{u} \in \mathcal{T}(G)$ the condition (a) together with the condition

(b*) $\kappa_p(G/G^*(\mathfrak{u})) = 0$ whenever $\mathfrak{u}(p) \neq \infty$

are fulfilled.

<u>Proof</u>. If G is completely decomposable and $G = \sum_{i \in I} J_i$ is a complete decomposition of G, then $\mathcal{T}(G)$ coincides also with the set of the types of all rank one groups J_i $(i \in I)$. Thus for any $\mathfrak{u} \in \mathcal{T}(G)$ the torsion free group $G(\mathfrak{u})/G^*(\mathfrak{u})$ is completely decomposable and homogeneous of the type $\mathfrak{u}$; evidently, $G(\mathfrak{u})/G^*(\mathfrak{u}) \in$

$\in \Gamma_\alpha$ $(1 \leqq \alpha \leqq 2)$. The group $G / G^*(\mathfrak{r})$ is completely decomposable as well and the types of its direct summands are $\leqq \mathfrak{r}$. Hence, if $\mathfrak{r}(p) \neq \infty$, then $G / G^*(\mathfrak{r})$ is p-reduced and in view of [4, Corollary 1] we have $r_p(G / G^*(\mathfrak{r})) = 0$. Thus in this case the conditions (a), (b*) are fulfilled.

Now, let us suppose that G satisfies (a) and (b*); we shall show that G is completely decomposable. From the hypothesis it follows immediately that $\mathcal{T}(G)$ is finite. Let us put $\mathcal{T}(G) = \{\mathfrak{r}_1 < \ldots < \mathfrak{r}_m\}$. Then we shall prove the complete decomposability of G by induction on $m = \operatorname{card} \mathcal{T}(G)$.

For $m = 1$ the group G is homogeneous of the type $\mathfrak{r}_1$ and $G^*(\mathfrak{r}_1) = 0$. Then the inequality $pG \neq G$ for a prime p implies $\mathfrak{r}_1(p) \neq \infty$ and in view of (b*) we have $0 = r_p(G / G^*(\mathfrak{r}_1)) = r_p(G)$. Hence, by [4, Corollary 4], G is completely decomposable.

Thus, suppose $m \geqq 2$ and let our assertion hold whenever the cardinality of the corresponding type set is $m-1$. Since $G(\mathfrak{r}_1) = G$, we can apply our Lemma to G for $\mathfrak{r} = \mathfrak{r}_1$ and we get

(1) $$G = H \dot{+} G^*(\mathfrak{r}_1),$$

where the group H is completely decomposable. If we put $G^*(\mathfrak{r}_1) = G_1 = G(\mathfrak{r}_2)$, then by (1) G_1 is also almost divisible and $\mathcal{T}(G_1) = \{\mathfrak{r}_2 < \ldots < \mathfrak{r}_m\}$. We shall now verify that G_1 fulfils (a) and (b*) for all types of $\mathcal{T}(G_1)$. In fact, if $\mathfrak{r} \in \mathcal{T}(G_1)$, then $\mathfrak{r}_1 < \mathfrak{r}_2 \leqq \mathfrak{r}$ and hence $G(\mathfrak{r}) \subseteq G(\mathfrak{r}_2) = G_1$, which implies

$G_1(\mu) = G(\mu)$; analogously, we obtain $G^*(\mu) \subseteq \subseteq G^*(\mu_1) = G_1$, therefore $G^*(\mu) = G_1^*(\mu)$. Thus we have $G_1(\mu)/G_1^*(\mu) = G(\mu)/G^*(\mu)$, which means that G_1 fulfils (a) for each $\mu \in \mathcal{T}(G_1)$. By (1), we can write for any $\mu \in \mathcal{T}(G_1)$

$$(2) \quad G/G^*(\mu) = (H \dotplus G_1)/G^*(\mu) \cong H \dotplus G_1/G^*(\mu) = \\ = H \dotplus G_1/G_1^*(\mu) ;$$

thus for $\mu(p) \neq \infty$ it is $\kappa_p(G \;/G^*(\mu)) = 0$ and hence by (2)

$$\kappa_p(H \dotplus G_1/G_1^*(\mu)) = 0 .$$

Following [4,Corollary 2], we get $\kappa_p(G_1/G_1^*(\mu)) = 0$, therefore the condition (b*) is satisfied by G_1 . Under the inductive hypothesis G_1 and in view of (1) G is completely decomposable as well. Thus the proof of our theorem is finished.

If the group G is torsion free of finite rank and H any of its pure subgroups, then $\kappa_p(G) = \kappa_p(H) + \kappa_p(G/H)$ for every prime p (see [6,Theorem 6]). In particular, we obtain that $\kappa_p(G) = 0$ implies $\kappa_p(G/H) = 0$ for each pure subgroup H of G . We shall use this last fact in the proof of the following theorem. Let us recall that if G is torsion free and p any prime, then $G[p^\infty]$ will denote the greatest p -divisible subgroup of G . Evidently, $\kappa_p(G[p^\infty]) = \kappa(G[p^\infty])$ (see [5,Theorem 1]).

<u>Theorem 2</u>. Let G be an almost divisible torsion free group of finite rank with the linearly ordered type set $\mathcal{T}(G)$. Then G is completely decomposable if and only if $\kappa_p(G) = \kappa(G[p^\infty])$ for every prime p .

Proof. If G is completely decomposable, then for any prime p, $G = G[p^{\infty}] \dotplus G_1$ where G_1 is also completely decomposable and p-reduced. Then, by [4, Corollary 1] $\varkappa_p(G_1) = 0$. Since $\varkappa_p(G) = \varkappa_p(G[p^{\infty}]) + \varkappa_p(G_1)$ (see [6, Theorem 6]), we get $\varkappa_p(G) = \varkappa_p(G[p^{\infty}]) = \varkappa(G[p^{\infty}])$.

To prove the converse consider $\varkappa_p(G) = \varkappa(G[p^{\infty}])$ for all primes p. For the proof of complete decomposability of G it suffices to show that G fulfils (b*) only, (a) being trivial. Let $\mathcal{T}(G) = \{\mathfrak{t}_1 < \ldots < \mathfrak{t}_n\}$, take $\mathfrak{t} \in \mathcal{T}(G)$ and suppose $\mathfrak{t}(p) \neq \infty$ for some prime p. In order to prove the relation $\varkappa_p(G/G^*(\mathfrak{t})) = 0$ we shall distinguish two cases: $G[p^{\infty}] = 0$ and $G[p^{\infty}] \neq 0$. If $G[p^{\infty}] = 0$, then $\varkappa_p(G) = \varkappa_p(G[p^{\infty}]) = 0$ and in view of the preceding remark we have $\varkappa_p(G/G^*(\mathfrak{t})) = 0$. If $G[p^{\infty}] \neq 0$, then there exists an integer $j \leqq n$ with $\mathfrak{t}_j(p) = \infty$; since $\mathfrak{t}_1 \leqq \mathfrak{t}$ and $\mathfrak{t}(p) \neq \infty$, it is certainly $1 < j$. Let i denote the smallest integer with $\mathfrak{t}_i(p) = \infty$; we shall show that $G[p^{\infty}] = G(\mathfrak{t}_i)$. The relation $\mathfrak{t}_i(p) = \infty$ implies the inclusion $G(\mathfrak{t}_i) \subseteq G[p^{\infty}]$. But if $0 \neq g \in G[p^{\infty}]$ and $\mathfrak{t}_k = \mathrm{type}(g)$, then $\mathfrak{t}_k(p) = \infty$, therefore $i \leqq k$. Hence we conclude $\mathfrak{t}_i \leqq \mathfrak{t}_k$ and $g \in G(\mathfrak{t}_i)$.

Thus we have shown that $G[p^{\infty}] = G(\mathfrak{t}_i)$ and also $G[p^{\infty}] = G^*(\mathfrak{t}_{i-1})$ $(2 \leqq i)$. By [6, Theorem 6] we have

$$\varkappa_p(G) = \varkappa_p(G[p^{\infty}]) + \varkappa_p(G/G[p^{\infty}]) ;$$

since $\varkappa_p(G[p^{\infty}]) = \varkappa(G[p^{\infty}]) = \varkappa_p(G)$, we get

$$(3) \quad 0 = \varkappa_p(G/G[p^{\infty}]) = \varkappa_p(G/G^*(\mathfrak{t}_{i-1})) .$$

From $\mathfrak{u}(p) \neq \infty$ it follows $\mathfrak{u} \not\leq \mathfrak{u}_{i-1}$ and hence $G^*(\mathfrak{u}_{i-1}) \subseteq G^*(\mathfrak{u})$. Thus we have

$$G/G^*(\mathfrak{u}) \cong (G/G^*(\mathfrak{u}_{i-1}))/(G^*(\mathfrak{u})/G^*(\mathfrak{u}_{i-1}))$$

and by (3) $r_p(G/G^*(\mathfrak{u})) = 0$. This means that G fulfils (b*) and Theorem 2 is proved.

Remark. The preceding theorem may be likewise formulated in the following way (see [2,Proposition 5.2]; for the definition of the regularity of a group see also [2, § 5]): Let G be an almost divisible torsion free group of finite rank with the linearly ordered type set $\mathcal{T}(G)$. Then the group G is completely decomposable if and only if it is regular.

Till now we have considered groups with the linearly ordered type set $\mathcal{T}(G)$ only. In order to investigate the general case we shall use [1,Theorem 48.6]. Thus we get the following assertion:

Theorem 3. An almost divisible torsion free group G is completely decomposable if and only if the conditions (a),(b) and

(c) $\quad G^*(\mathfrak{u}) = G(\mathfrak{u}) \cap G^{**}(\mathfrak{u})$

are fulfilled for each type $\mathfrak{u}$ $\mathcal{T}(G)$.

Proof. Firstly, assume that G is completely decomposable and that $G = \sum_{\lambda \in \Lambda} J_\lambda$ is one of its complete decompositions. Denote by $T(G)$ the set of all types of the groups $J_\lambda (\lambda \in \Lambda)$; evidently, $T(G) \subseteq \mathcal{T}(G)$. For $\mathfrak{u} \in T(G)$ let $A_{\mathfrak{u}}$ denote the direct sum of all groups J_λ of the type $\mathfrak{u}$; certainly, it is $G(\mathfrak{u})/G^*(\mathfrak{u}) \cong A_{\mathfrak{u}}$. If $\mathfrak{u}(p) \neq \infty$, then $A_{\mathfrak{u}}$ is a p-reduced completely decomposable group and in view of [4,Corollary 1] $0 =$ $= \kappa_p(A_{\varkappa}) = \kappa_p(G(\varkappa)/G^*(\varkappa))$. Thus for $\varkappa \in T(G)$ the conditions (a) and (b) are fulfilled. But if $\varkappa \in \mathcal{T}(G) \dot{-}$ $\dot{-} T(G)$, then $G(\varkappa) = G^*(\varkappa)$ and the conditions (a), (b) are trivial. The condition (c) follows from [1,Theorem 48.6].

Further, suppose that G fulfils the conditions (a), (b),(c), and prove that G is completely decomposable. If $\varkappa \in \mathcal{T}(G)$, then by Lemma there exists a direct decomposition of the form $G(\varkappa) = G_{\varkappa} \dot{+} G^*(\varkappa)$ where the group $G_{\varkappa}$ is completely decomposable and homogeneous of the type $\varkappa$. Now, the proof proceeds in the same way as that of sufficiency in [1,Theorem 48.6]. Thus, firstly, it may be shown that the subgroups $G_{\varkappa}$ ($\varkappa \in \mathcal{T}(G)$) generate their direct sum $\sum_{\varkappa} G_{\varkappa}$, and then we should get $G = \sum_{\varkappa} G_{\varkappa}$. The last relation is proved in [1,Theorem 48.6] under the assumption that $\mathcal{T}(G)$ satisfies the maximum condition, but in our case $\mathcal{T}(G)$ is finite, G being almost divisible. Since each $G_{\varkappa}$ is completely decomposable, so is the group $G = \sum_{\varkappa} G_{\varkappa}$ as well, which finishes the proof of the theorem.

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