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# EVERY GROUP IS A MAXIMAL SUBGROUP OF THE SEMIGROUP OF RELATIONS

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The aim of this note is to extend a result of [2], namely to prove the following theorem:

Theorem: The class of maximal subgroups of semigroups of binary relations includes all groups.

This generalizes [2], Theorem 4.7 to infinite groups.x)
We preserve the notation of [2] and refer to the results proved there, too.
Concerning graphs we use the notation of [1].

Proof of the theorem: Let $G$ be an infinite group (the proof for finite case would be similar; since the finite case is solved in [2], we make this assumption for sake of brevity). By [1], there is a graph $(X, R)$ such that $C(X, R) \simeq G$, where $C(X, R)$ is the monoid of all compatible mappings (i.e. homomorphisms) into itself. By constructions given in [1], we can assume the following about the graph $(X, R)$:

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x) Using a different method this generalization was obtained independently by A.H. Clifford, R.J. Plemmons and B.M. Schein.

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a) $|X| = |R|$ (this follows from the fact that $(X, R)$ can be chosen without isolated points).

b) Let $V(x) = \{y \mid (x,y) \in R\}$, then $x \neq y$ implies $V(x) \nsubseteq V(y)$ and $V(y) \nsubseteq V(x)$. Similarly for $\bar{V}(x) = \{y \mid (y,x) \in R\}$.

c) $V(x) \neq \emptyset$, $V(x) \neq X$ for every $x \in X$. Similarly for $\bar{V}(x)$.

Let $\varphi : X \longrightarrow R$ be a bijection. Define the relation $\alpha$ on $X_{01} = X \times \{0, 1\}$ ($0,1 \notin X$) by:

$$((x,0),(y,0)) \in \alpha \iff ((x,1),(y,1)) \in \alpha \iff x = y,$$

$((x,0),(y,1)) \in \alpha \iff x$ is incident with $\varphi(y)$,

$$((x,1),(y,0)) \notin \alpha .$$

By b),c), $\alpha$ is reduced. Further, $\alpha$ is idempotent as can be easily seen. Thus by Lemma 3.4 [2] (and by its remark), the maximal subgroup $H_\alpha$ of $\mathcal{B}_X$ containing $\alpha$ is given by $H_\alpha \simeq G_\alpha = \{\varphi \in S_{X_{01}} \mid \exists \sigma \in S_{X_{01}}\ \alpha\varphi = \sigma\alpha\}$. But in this special case we have $G_\alpha = \{\varphi \mid \alpha\varphi = \varphi\alpha\}$. Similarly as in the proof of [2], Lemma 4.2,
$G_\alpha \simeq \{\varphi \in S_X \mid \exists \sigma \in S_X,\ R\varphi = \sigma R\} = G_R$.
But obviously $G_R = A(X,R) = C(X,R) \simeq G$, by the assumption ($A(X,R)$ is the group of all automorphisms of the graph $(X,R)$).

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References

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