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SCHWARTZ SPACES CONSISTENT WITH A DUALITY

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The purpose of this note is to state a proposition characterizing topologies on E, making E a Schwartz space, as topologies of uniform convergence on certain subsets of $E' = F$. There exists always the weakest and the strongest of such topologies. The propositions are quite analogous to those in [1]. All the statements of [1] remain valid, if the terms of the nuclear space and of the hypernuclear set are changed to the terms of the Schwartz space and of the (S)-set (see the definition below), respectively. We rewrite here for Schwartz spaces only the main propositions. All the proofs are very easy and are omitted.

By a neighbourhood $\mathcal{U}$ of zero in a locally convex vector space E we mean a closed absolutely convex neighbourhood, by $E_{\mathcal{U}}$ we denote the normed space obtained from E by taking $\mathcal{U}$ as the unit ball and passing to a factor space and if $\mathcal{U} \subset V$ are neighbourhoods of zero, then $E(\mathcal{U}, V)$ means the continuous map $E_{\mathcal{U}} \longrightarrow E_V$ obtained from the identity transformation of E (see [3, 0.11]).

<u>1. Definition</u>. Let E be a locally convex space. A neighbourhood $\mathcal{U}$ of zero in E is called (S)-neighbourhood, if there exists a sequence $\{\mathcal{U}_n\}$ of neighbourhoods

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of zero in E, such that $\mathcal{U}_0 = \mathcal{U}$, $\mathcal{U}_{n+1} \subset \mathcal{U}_n$ and $E(\mathcal{U}_{n+1}, \mathcal{U}_n)$ is completely continuous for all n. A set $B \subset E'$ is called (S)-set, if there is a (S)-neighbourhood $\mathcal{U}$ of zero in E, such that $B \subset \mathcal{U}^\circ$.

This concept evidently depends on the topology $\wp$ of E; we also write $\wp-(S)$-set.

2. Proposition. a) E is a Schwartz space if and only if E has a basis of neighbourhoods of zero consisting of (S)-neighbourhoods.

b) E is a Schwartz space if and only if every equicontinuous subset of E' is a (S)-set.

c) Let $\mathcal{F}$ be a full family (see [1]) of $\wp-(S)$-subsets of E'. Let $\wp_{\mathcal{F}}$ be a topology on E of uniform convergence on the elements of $\mathcal{F}$. Then $(E, \wp_{\mathcal{F}})$ is a Schwartz space and has the same dual as $(E, \wp)$. $\wp_{\mathcal{F}}$ is the finest topology of a Schwartz space on E coarser than $\wp$.

3. Proposition. Let $\langle E, F \rangle$ be a duality and $\tau(E,F)$ the Mackey topology on E of the pair $\langle E, F \rangle$. A topology $\wp$ on E is consistent with the duality and makes E a Schwartz space if and only if $\wp$ is the topology of uniform convergence on some full family of $\tau(E,F)-(S)$-subsets of F.

If we take the set of all $\tau(E,F)-(S)$-subsets of for $\mathcal{F}$, then $\wp_{\mathcal{F}}$ is the finest topology of a Schwartz space on E having F as its dual space. The weak topology $\sigma(E,F)$ is evidently the weakest of such topologies.

Remark. In [1] it was shown that not every prenuclear set is hypernuclear. Here the parallel is to ask whether it is not sufficient in the definition 1 to demand only the existence of one $\mathcal{U}_1$ instead of the sequence $\{\mathcal{U}_n\}$. Probably, this reduced definition is sufficient, due to different properties of compact and nuclear maps, namely that every compact map can be expressed as the product of two compact maps, which is not true in the category of nuclear maps.

References

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