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ON NORMS AND SUBSETS OF LINEAR SPACES

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J. Zemánek has given [10] an example of a non-empty finitely open and nowhere dense convex subset of a normed linear space. Some general theorems concerning the existence of comparable non-equivalent norms in infinite-dimensional spaces give a possibility to construct simpler examples of that type (see Proposition 1 and Examples 1 - 3 below).

Throughout this paper, X denotes a real linear space. Let G be a subset of X. G is said to be: (1) finitely open (see [6], Definition 1.10.2) if each finite-dimensional affine subspace L of X intersects G in a set open in L (in the unique linear topology on L), (2) linearly bounded if its intersection with any line is bounded (as a subset of the line). The convex hull of G is denoted by $conv\, G$; $diam_{\|\cdot\|}\, G$ denotes the diameter of G in $(X, \|\cdot\|)$, where $\|\cdot\|$ is a norm on X, "$\xrightarrow{\|\cdot\|}$" denotes the convergence in the topology given by $\|\cdot\|$. G is said to be $\|\cdot\|$-P if G is P in $(X, \|\cdot\|)$ where P is a property of subsets of X (we shall use P = weak, bounded, open). G is a convex body if it is convex and has a

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non-empty interior in $(X, \|\cdot\|)$.

We begin with

Proposition 1. Let $\|\cdot\|_0$ and $\|\cdot\|_1$ be two non-equivalent norms on a linear space X such that $\|\cdot\|_0 \leq K\|\cdot\|_1$ (for some $K > 0$). Then $C = \{x \in X : \|x\|_1 < 1\}$ is a finitely open nowhere dense absolutely convex (non-empty) subset of $(X, \|\cdot\|_0)$. Clearly, X must be infinite-dimensional.

Proof. Clearly, C is absolutely convex and non-empty. Since C is open in $(X, \|\cdot\|_1)$ it is finitely open. Let C_0 denote the closure of C in $(X, \|\cdot\|_0)$. For each $y \in C_0$ there is $x_y \in C$ such that $\|y - x_y\|_0 < 1$. Then

$$\|y\|_0 \leq \|y - x_y\|_0 + \|x_y\|_0 \leq \|y - x_y\|_0 + K\|x_y\|_1 < 1 + K.$$

Hence $C_0 \subset (K+1)C$. Suppose that C_0 has a non-empty interior in $(X, \|\cdot\|_0)$. Then the absolute convexity of C_0 implies the existence of some $k > 0$ such that $\{x \in X : \|x\|_0 < k\} \subset C_0$. This and $C_0 \subset (K+1)C$ imply that $\|\cdot\|_1 \leq k^{-1}(K+1)\|\cdot\|_0$, a contradiction to the non-equivalence of both norms.

Proposition 2. Let $\|\cdot\|_0$ and $\|\cdot\|_1$ be two norms on a linear space X such that $\|\cdot\|_0 \leq K\|\cdot\|_1$ $(K > 0)$. Define $\|\cdot\|_t = (1-t)\|\cdot\|_0 + t\|\cdot\|_1$ for $0 \leq t \leq 1$. Then 1° $\|\cdot\|_t$, $t \in [0,1]$ are the norms on X, 2° $\|\cdot\|_{t_1} \leq K(t_1, t_2)\|\cdot\|_{t_2}$ for $0 \leq t_1 \leq t_2 \leq 1$, where $K(t_1, t_2) = [t_1 + K(1-t_1)][t_2 + K(1-t_2)]^{-1}$, 3° $\|\cdot\|_{t_2} \leq t_2 t_1^{-1}\|\cdot\|_{t_1}$ for $0 < t_1 \leq t_2 \leq 1$, and hence the norms $\|\cdot\|_{t_1}$ and $\|\cdot\|_{t_2}$ are equivalent, 4° if the norms $\|\cdot\|_0$ and $\|\cdot\|_1$ are non-equivalent, then $\|\cdot\|_0$ and $\|\cdot\|_t$, $t \in (0,1]$, are non-equivalent.

The proof goes by a direct computation.

Proposition 2 says that two comparable norms can be joined by a "continuum" of pairwise equivalent norms.

The following two theorems were first proved in our thesis [3] and published without proof in [4].

Theorem 1. Let $(X, \|\cdot\|)$ be a normed linear space such that its dual space X^* is separable. Then there exists a norm $\|\cdot\|_w$ on X such that the $\|\cdot\|$-weak topology and the $\|\cdot\|_w$-topology coincide on the $\|\cdot\|$-bounded subsets of X, and $\|\cdot\|_w \leqq \|\cdot\|$. If X has an infinite dimension, then the norms $\|\cdot\|_w$ and $\|\cdot\|$ are non-equivalent.

Proof. Let $\{u_n\}$ be a dense sequence in the unit ball of X^* and $\|x\|_w = \sum_{n=1}^{\infty} 2^{-n} |u_n(x)|$ for x in X. It is easy to see that $\|\cdot\|_w$ is a norm and $\|\cdot\|_w \leqq \|\cdot\|$. Let M be a $\|\cdot\|$-bounded subset of X, x_0 a point of M. If W is a weak neighbourhood of x_0 in M then there exist $\varepsilon > 0$ and $f_1, \dots, f_m \in X^*$, $\|f_j\| = 1$ $(j = 1, \dots, m)$ such that $W_1 = \{x \in M : |f_j(x - x_0)| < \varepsilon$ for $j = 1, \dots, m\} \subset W$. Clearly, W_1 is a weak neighbourhood of x_0 in M. Without loss of generality we may suppose that M contains at least two points. There are integers $n_1, \dots, n_m$ such that $\|u_{n_j} - f_j\| < \varepsilon (4 \operatorname{diam}_{\|\cdot\|} M)^{-1}$ for $j = 1, \dots, m$. Let $N = 1 + \max\{n_1, \dots, n_m\}$ and $V = \{x \in M : \|x - x_0\|_w < \varepsilon 2^{-N}\}$. We shall show that $W_1 \supset V$. Let $x \in V$. Then $2^{-n_j} |(u_{n_j} - f_j)(x - x_0) + f_j(x - x_0)| \leqq \|x - x_0\|_w < \varepsilon 2^{-N} \leqq \frac{\varepsilon}{2} 2^{-n_j}$ for $j = 1, \dots, m$. Since $|(u_{n_j} - f_j)(x - x_0)| \leqq \|u_{n_j} - f_j\| \|x - x_0\| < \varepsilon/4$, there is $|f_j(x - x_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} < \varepsilon$ for $j = 1, \dots, m$. Hence $x \in W_1$ and $V \subset W_1 \subset W$. Conversely, let $V = \{x \in M :$

$: \|x-x_0\|_w < \varepsilon\}$ $(\varepsilon > 0)$ be a $\|\cdot\|_w$-neighbourhood of x_0 in M. A direct calculation shows that V contains $W = \{x \in M : \sum_{n=1}^{m} 2^{-n} |u_n(x-x_0)| < \varepsilon/2\}$ where m is so large that $\sum_{n=m+1}^{\infty} 2^{-n+1}\, diam\, M < \varepsilon$. Clearly, W is a $\|\cdot\|$-weak neighbourhood of x_0 in M. Suppose that X is infinite-dimensional and the norms $\|\cdot\|_w$ and $\|\cdot\|$ are equivalent. Let us denote $X^* = (X, \|\cdot\|)$ and $B = \{x \in X : \|x\| \leq 1\}$. Then

$$\begin{array}{ccc} (B, \|\cdot\|) & \xrightarrow{\ id\ } & (B, \sigma(X, X^*)) \\ & {\scriptstyle id} \nwarrow\!\searrow \quad \swarrow {\scriptstyle id} & \\ & (B, \|\cdot\|_w) & \end{array}$$

is a commutative diagram of topological spaces and continuous mappings; (B, τ) denotes the set B with the topology induced by the τ-topology of X. Thus, the three topologies $\|\cdot\|$, $\|\cdot\|_w$, and $\sigma(X, X^*)$ coincide on B, a contradiction to the infinite dimensionality of X (see [5], Chapt.V, Exerc. 7.9). Hence the norms $\|\cdot\|_w$ and $\|\cdot\|$ are non-equivalent. The proof is complete.

<u>Theorem 2</u>. Let $(X, \|\cdot\|)$ be a separable normed linear space. Then there exists a norm $\|\cdot\|_w$ on X such that the $\|\cdot\|$-weak topology is on $\|\cdot\|$-bounded subsets of X stronger than the $\|\cdot\|_w$-topology, and $\|\cdot\|_w \leq \|\cdot\|$. If X has infinite dimension, then the norms $\|\cdot\|_w$ and $\|\cdot\|$ are non-equivalent.

<u>Proof</u>. By [1], Chapt.III, Theorem 9.16 the unit ball of X^* contains a sequentially $\sigma(X^*, X)$-dense sequence

$(X^* = (X, \|\cdot\|)^*)$; let $S = \{u_n\}$ be such a sequence, and set $\|x\|_w = \sum_{n=1}^{\infty} 2^{-n} |u_n(x)|$ for x in X. Let $0 \neq x \in X$. Then there is $f \in X^*$ such that $|f(x)| = \varepsilon > 0$. Since $\mathbb{R}S = \{r u_n : r \in \mathbb{R}, n = 1, 2, \ldots\}$ is $\sigma(X^*, X)$ -dense in X^*, there exist $r \in \mathbb{R}$ and u_n such that $r u_n$ lies in the $\sigma(X^*, X)$ -neighbourhood $\{x^* \in X^* : |(x^* - f)(x)| < \varepsilon\}$ of f. Then $|r u_n(x)| \geq |f(x)| - |(r u_n - f)(x)| > 0$. Hence $\|x\|_w > 0$, and $\|\cdot\|_w$ is a norm on X. The proof of the other assertions of the theorem is the same as that of the corresponding assertions of Theorem 1.

Theorem 3 below is the precise statement of the results of the proof of Proposition 1.1 in [2]. That proof relies on a paper of V. Klee [7]. We repeat their proof making use of Theorem 2 instead of [7].

<u>Theorem 3</u>. Let $(X, \|\cdot\|)$ be an infinite-dimensional normed linear space. Then there are two norms $|\cdot|$ and $|||\cdot|||$ on X such that $|\cdot| \lneq \|\cdot\| \lneq |||\cdot|||$ and none of them is equivalent to $\|\cdot\|$. If $\|\cdot\|$ is complete (that is, $(X, \|\cdot\|)$ is complete), the norms $|\cdot|$ and $|||\cdot|||$ are not.

<u>Proof</u>. Let B be a Hamel basis for X such that $\|b\| \leq 1$ for all $b \in B$ and $\inf\{\|b\| : b \in B\} = 0$. It is easy to verify that $|||\cdot|||$ defined as the Minkowski functional of the absolutely convex hull of B, satisfies our requirements.

Let L be a separable infinite-dimensional subspace of $(X, \|\cdot\|)$, $\|\cdot\|_w$ the norm of Theorem 2 corresponding to $(L, \|\cdot\|)$, and $V = \{x \in L : \|x\|_w \leqq 1\}$. By Theorem 2, the norms $\|\cdot\|_w$ and $\|\cdot\|$ on L are non-equivalent and $\|\cdot\|_w \lneqq \|\cdot\|$. This implies that the set V is unbounded in $(L, \|\cdot\|)$; V is linearly bounded since it is bounded in $(L, \|\cdot\|_w)$. Hence V is an absolutely convex, linearly bounded, unbounded closed body in $(L, \|\cdot\|)$. Let $U = \{x \in X : \|x\| \leqq 1\}$. Then $C = \mathrm{co}(U \cup V)$ is an absolutely convex body in $(X, \|\cdot\|)$. Suppose that C is not linearly bounded. Then C contains a line J through 0. Let $x \in J$. For each integer n, $nx \in J$ and hence $nx = \lambda_n x_n + (1 - \lambda_n) y_n$ for some $\lambda_n \in [0,1]$, $x \in U$, $y_n \in V$. Since $n^{-1} \lambda_n x_n \xrightarrow{\|\cdot\|} 0$, we have $V \ni$ $\ni n^{-1}(1 - \lambda_n) y_n \xrightarrow{\|\cdot\|} x$. V is $\|\cdot\|$-closed and hence $x \in V$. This implies that $J \subset V$, a contradiction to the linear boundedness of V. We have proved that the set C must be linearly bounded. Hence its Minkowski functional $|\cdot|$ defines a norm for X. The inclusion $U \subset C$ implies $|\cdot| \leqq \|\cdot\|$. Since C is unbounded in $(X, \|\cdot\|)$, the norms $|\cdot|$ and $\|\cdot\|$ are non-equivalent.

The part of the theorem concerning the completeness follows from the open mapping theorem.

<u>Theorem 4</u>. Let X be an infinite-dimensional linear space and C a non-empty absolutely convex, linearly bounded, finitely open subset of X. Then there are two norms $|\cdot|$ and $\|\cdot\|$ on X such that C is open in $(X, \|\cdot\|)$ and nowhere dense in $(X, |\cdot|)$ and $|\cdot| \lneqq \|\cdot\|$.

<u>Proof</u>. Let $\|\cdot\|$ be the Minkowski functional of C. It is a norm on X. It is sufficient to use Theorem 3 and

then Proposition 1.

Theorem 5. Let $(X, \|\cdot\|)$ be a normed linear space of infinite dimension. Then there is a non-empty absolutely convex finitely open bounded and nowhere dense subset C of $(X, \|\cdot\|)$.

Proof. Let $|||\cdot|||$ be as in Theorem 3. It is sufficient to set $C = \{x \in X : |||x||| < 1\}$ and apply Proposition 1.

Corollary. Let X be an infinite-dimensional linear space. Then:

1. there is neither a minimal nor maximal norm on X (a norm $\|\cdot\|$ on X is said to be minimal [maximal] if for any norm $|||\cdot|||$ on X there exists $K > 0$ such that $\|\cdot\| \leq K\,|||\cdot|||$ $[K\,|||\cdot||| \leq \|\cdot\|]$);

2. the strongest locally convex topology on X is not normable;

3. if (X, τ) is a locally convex space of minimal type (see [9], Chapt. IV, Exerc. 6), it is non-normable.

Remark. Any finitely open convex subset of X is open in the strongest locally convex topology on X. Hence there is no finitely open non-empty convex subset of X which is nowhere dense in the strongest locally convex topology. The second part of our corollary is not the best possible result; see [9], Chapt. II, Exerc. 7.

Examples. 1. Let G be a compact subset of $\mathbb{R}^n$ $(n \geq 1)$ with a positive Lebesgue measure, $\operatorname{mes} G > 0$, X the linear space of all continuous real-valued functions on G, $\|\cdot\|$ the sup norm on X, $|\cdot| = \|\cdot\|_{L_p(G)}$ $(p \geq 1)$. Then

$|\cdot| \leqq \|\cdot\|$ and these norms are non-equivalent on X. (Hint: For any $\varepsilon > 0$, there exist disjoint closed subsets M_ε, N_ε of G such that $0 < mes\, M_\varepsilon < \varepsilon$, $mes\, N_\varepsilon > mes\, G - 2\varepsilon$. Let $u_\varepsilon \in X$ be such that $u_\varepsilon | M_\varepsilon = (2\varepsilon)^{-1/p}$, $u_\varepsilon | N_\varepsilon = 0$, $0 \leqq u_\varepsilon \leqq (2\varepsilon)^{-1/p}$. Then $\| u_\varepsilon \| = (2\varepsilon)^{-1/p}$, $|u_\varepsilon| = (\int_{G \setminus N_\varepsilon} |u_\varepsilon(x)|^p dx)^{1/p} \leqq$ $\leqq (2\varepsilon \cdot (2\varepsilon)^{-1})^{1/p} = 1$.

Another hint: If both norms are equivalent on $X = C(G)$, then $C(G)$ is a closed and dense subspace of $L_p(G)$. This leads to a contradiction.) By Proposition 1, $C = \{x \in X ; \|x\| < 1\}$ is a finitely open, absolutely convex, nowhere dense, bounded (non-empty) subset of $(X, |\cdot|)$.

2. Let G be as above and $1 \leqq p'' < p < p' \leqq \infty$. Set

$X = L_{p'}(G)$, $\|\cdot\| = \|\cdot\|_{L_p(G)}$, $|||\cdot||| = (mes\, G)^{1/p - 1/p'} \|\cdot\|_{L_{p'}(G)}$,

and $|\cdot| = (mes\, G)^{1/p - 1/p''} \|\cdot\|_{L_{p''}(G)}$. Then $|\cdot| \leqq \|\cdot\| \leqq$ $\leqq |||\cdot|||$. Any two of these norms are non-equivalent on X. (Hint: By [8], § 12, Sect. 1, we may restrict ourselves to the easy case $G = [0,1]$.)

3. Let

$1 \leqq p'' < p < p' \leqq \infty$, $X = \ell_{p''}$, $\|\cdot\| = \|\cdot\|_{\ell_p}$, $|\cdot| = \|\cdot\|_{\ell_{p'}}$,

and $|||\cdot||| = \|\cdot\|_{\ell_{p''}}$. Then $|\cdot| \leqq \|\cdot\| \leqq |||\cdot|||$ and any two of these norms are non-equivalent.

<u>Remark</u>. Does Theorem 4 hold with "absolutely convex" replaced by "convex" ? This leads to another question. Is the absolute convex hull of a convex linearly bounded finitely open set linearly bounded? We conjecture that the answer is (generally) no.

If $\|\cdot\|_0$ and $\|\cdot\|_1$ in Proposition 2 are non-equivalent, does there exist a "monotone continuum" of pairwise non-equivalent comparable norms ? The answer is yes, when $\|\cdot\|_i$ $(i = 0, 1)$ are the L_{p_i}-norms on $X = L_{p_1}$ $(p_0 < p_1)$ or the ℓ_{p_i}-norms on $X = \ell_{p_1}$ $(p_0 > p_1)$.

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