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A NOTE ON RUDIN'S EXAMPLE OF DOWKER SPACE

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One assumption on a topological space occurs very frequently in mathematics - the property of being normal and countably paracompact. E.g., in such a space every Baire measure can be extended to a Borel measure ([3]). In a normal and countably paracompact space, the realcompactness can be described without (explicit or implicit) use of the notion of zero-set (see [1], Note 3), namely a closed-complete space (defined below) is realcompact.

Last year Mrs. M.E. Rudin gave an example of a normal Hausdorff space Y which is not countably paracompact ([4],[5]). It seemed quite natural to study some other properties of the space in order to show the importance of the assumption of countable paracompactness in the theorems above. It will be proved in the present note that the space Y is closed-complete not being realcompact, almost realcompact nor Baire-Borel complete (definitions below).

The following theorem was communicated to me by Z. Frolík:

Let P be a normal closed-complete topological space. Then the following conditions are equivalent:

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(a) P is realcompact.

(b) Let $\mathcal{F}$ be a maximal centered collection of closed sets without a countable intersection property. Then the collection $\mathcal{Z} = \{ Z \in \mathcal{F} \mid Z$ is a zero-set in $P \}$ has not the countable intersection property.

We shall see that the space Y serves as a counter-example - that both conditions (a) and (b) are not sufficient.

Let us recall the Rudin's definitions:

If λ is an ordinal, let $L(\lambda)$ denote the set of all ordinals less than λ and let $K(\lambda)$ denote the cofinality of λ. Let N denote the set of all positive integers.

For all $n \in N$, we define an ordinal λ_n by induction. Define λ_1 to be the smallest ordinal such that $K(\lambda_1)$ is greater than the cardinality of continuum. And, if λ_n has been defined, define λ_{n+1} to be the smallest ordinal such that $K(\lambda_{n+1}) > K(\lambda_n)$.

Let λ be the limit of $\{ \lambda_n \mid n \in N \}$.

Let $X = \{ f : N \to L(\lambda) \mid \forall n \in N \ f(n) \leq \lambda_n \}$.

Let $Y = \{ f \in X \mid \forall n \in N, K(f(n)) > \aleph_0$, but $\exists k \in N$ such that $\forall n \in N, K(f(n)) < K(\lambda_k) \}$.

Suppose f and g belong to X. If, for all $n \in N$, $f(n) < g(n)$, we say $f < g$. If, for all $n \in N$, $f(n) \leq g(n)$, we say $f \leq g$.

Let $\mathcal{B} = \{ U \subset Y \mid$ for some $f \in X$ and $g \in Y$, $U = \{ y \in Y \mid f < y \leq g \} \}$.

The system $\mathcal{B}$ is a basis for a topology on Y. This space Y is Hausdorff, normal and has the Dowker property, i.e. there exists a decreasing sequence of closed sets D_n in Y with a void intersection, but, for any open $U_n \supset D_n$ the intersection $\bigcap \{U_n \mid n \in N\}$ is non-void.

We may endow the set X with a suitable topology: $U \subset X$ is open iff for every $x \in U$ there exists a $y \in X$ such that the set

$\{z \in X \mid k \in N, x(k) > 0$ implies $y(k) < z(k) \leq x(k)$;

$k \in N, x(k) = 0$ implies $y(k) = z(k) = x(k)\}$ is a subset of U .

Then Y is a subspace of X .

Denote $X' = \{f \in X \mid \forall n \in N, f(n) < \lambda_n\}$ and $\mathcal{A} = \{A_f \mid f \in X'\}$, where $A_f = \{g \in Y \mid g > f\}$. It is easy to verify that the system $\mathcal{A}$ has the countable intersection property (abbr. CIP) and that all A_f are clopen.

<u>Lemma 1</u>. Let $\mathcal{C} \supset \mathcal{A}$ be a filter in the space Y and let $C \cap X'$ be non-void whenever $C \in \mathcal{C}$. Then the system $\{\overline{C} \mid C \in \mathcal{C}\}$ has CIP. Moreover, $\bigcap \{\overline{C}_i \mid i \in N, C_i \in \mathcal{C}\} \cap X'$ is non-void. (The symbol $\overline{C}$ denotes the closure of C in Y .)

<u>Proof</u>. Given $\{C_i \mid i \in N\} \subset \mathcal{C}$, we may assume that $C_1 \supset C_2 \supset \dots$. The proof goes by transfinite induction:

I. Let us choose some $f_0 \in C_1 \cap X'$.

II. Let $\alpha < \omega_1$ and suppose that $f_\beta \in Y \cap X'$ are defined for all $\beta < \alpha$.

a) Suppose $\alpha = \beta + 1$. Let j be the smallest index such that $f_\beta \notin C_j$. Because of the condition on $\mathcal{C}$ we can find $f_\alpha \in C_j \cap A_{f_\beta} \cap X'$.

b) Suppose α is a limit ordinal. Let $g_\alpha(n) =$ $= \sup\{f_\beta(n) \mid \beta < \alpha\}$ for all $n \in N$. Since $g_\alpha \in X'$, we can find $f_\alpha \in C_1 \cap A_{g_\alpha} \cap X'$.

Define $f(n) = \sup\{f_\alpha(n) \mid \alpha < \omega_1\}$. Since the sequence $\{f_\alpha \mid \alpha < \omega_1\}$ is increasing, $K(f(n)) = \omega_1$ for all $n \in N$, which implies that $f \in Y \cap X'$.

It remains to show that $f \in \bigcap\{\overline{C}_i \mid i \in N\}$. Fix $i \in N$ and $g < f$, $g \in X$. For every $n \in N$ there exists f_{α_n} with $f_{\alpha_n}(n) > g(n)$. Put $\beta = \sup\{\alpha_n \mid n \in N\}$. From the definition of f_α there follows easily that $f_{\beta+i} \in$ $\in C_i$, $g < f_{\beta+i} \leq f$. Since g was chosen arbitrarily, $f \in \overline{C}_i$.

<u>Corollary 1</u>. Let $\mathcal{C}$ be an ultrafilter in Y, $\mathcal{C} \supset$ $\supset \mathcal{A} \cup \{Y \cap X'\}$. Then the system $\{\overline{C} \mid C \in \mathcal{C}\}$ has CIP.

The ultrafilter $\mathcal{C}$ has a void intersection, because $\bigcap \mathcal{A} = \emptyset$.

<u>Definition</u> ([1],[2]). Let P be a topological space.

A space P is called <u>almost realcompact</u>, if for each maximal centered family $\mathcal{U}$ of open sets, such that $\overline{\mathcal{U}}$ has the countable intersection property, $\bigcap \overline{\mathcal{U}}$ is non-void.

A space P is called <u>closed-complete</u>, if for each maximal centered family $\mathcal{F}$ of closed sets with CIP $\bigcap \mathcal{F}$ is non-void.

A space P is called <u>Baire-Borel complete</u>, if a maximal centered collection $\mathcal{Z}$ of zero-sets with CIP has a non-void intersection whenever there exists some maximal centered collection $\mathcal{B}$ of Borel sets with CIP such that $\mathcal{B} \supset \mathcal{Z}$.

Since it follows from Corollary 1 that $\mathcal{Y}$ is not almost realcompact and since realcompact implies almost realcompact, we have the following

Corollary 2. The space $\mathcal{Y}$ is neither almost realcompact nor realcompact.

Corollary 3. The space $\mathcal{Y}$ is not Baire-Borel complete.

Proof. Let $\mathfrak{Z}$ be a maximal centered collection of zero-sets in $\mathcal{Y}$, $\mathfrak{Z} \supset \mathcal{A}$. Then there exists at least one system $\mathcal{H}$ of closed sets in $\mathcal{Y}$, $\mathcal{H} \supset \mathfrak{Z}$, such that $\mathcal{F} = \{H \cap X' \mid H \in \mathcal{H}\}$ is centered and maximal (to see this one must show that $Z \in \mathfrak{Z}$ implies $Z \cap X' \neq \emptyset$ and the rest is routine. But it is easy to prove that each closed set Z in $\mathcal{Y}$ disjoint with X' cannot be G_δ - a contradiction.) According to Lemma, $\mathcal{F}$ is countably centered. Let $\mathcal{B} \supset \mathcal{F}$ be a maximal centered collection of Borel sets in $\mathcal{Y}$. $\mathcal{F}$ is a base for $\mathcal{B}$ (this is left to the reader), which implies that $\mathcal{B}$ is countably centered, too. From the maximality of $\mathcal{B}$ we have $\mathcal{B} \supset \mathfrak{Z}$. Since $\mathfrak{Z} \supset \mathcal{A}$, $\cap \mathfrak{Z}$ is void.

Theorem 1. [x)] There exists a maximal centered collection $\mathfrak{Z}$ of zero-sets in $\mathcal{Y}$ with CIP which cannot be extended to the maximal centered collection of closed sets

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x) A collection of zero-sets with the same properties can be found also in Mrówka's example of non-realcompact almost realcompact space, but Mrówka's space is non-normal. It was stated in

S. Mrówka: On the union of Q-spaces, Bull.Acad.Polon.Sci., 6(1958),365-367;

S. Mrówka: Some comments on the author's example of non-$\mathcal{R}$-compact space, ibid., 18(1970),443-448.

in Y with CIP.

Proof. Let $\mathcal{Z} \supset \mathcal{A}$ be a maximal centered collection of zero-sets. We know that it is countably centered. Let $\mathcal{F} \supset \mathcal{Z}$ be a maximal centered collection of closed sets.

Let $D_n = \{y \in Y \mid \exists k \geq n,\ y(k) = \lambda_k\}$.

In [4] it was shown that $\bigcap\{D_n \mid n \in N\} = \emptyset$ and that D_n are closed. We have to prove that $D_n \in \mathcal{F}$, which means that $\mathcal{F}$ is not countably centered.

Suppose, for some $i \in N$, $D_i \notin \mathcal{F}$. Then there exists an $F \in \mathcal{F}$ disjoint with D_i. Since Y is normal, there is a continuous real-valued function $\psi : Y \to R$, $\psi[F] \subset (0)$, $\psi[D_i] \subset (1)$. The continuity of ψ implies that $\mathcal{U} = \psi^{-1}[\,]1/2, 3/2[\,]$ is an open neighborhood of D_i, disjoint with F. It was proved in [4] that there exists an $f \in Y \cap X'$, such that $\{y \mid y > f\} \subset \mathcal{U}$. Thus $\emptyset = \mathcal{U} \cap F \supset A_f \cap F$, which is a contradiction with $\mathcal{F} \supset \mathcal{A}$.

Theorem 2. The space X is realcompact.

Proof. Denote $K_{n,\alpha} = \{f \in X \mid f(n) > \alpha\}$, $L_{n,\alpha} = \{f \in X \mid f(n) \leq \alpha\}$ for $\alpha \leq \lambda_n$. $K_{n,\alpha}, L_{n,\alpha}$ are clopen in X, hence zero-sets.

Let $\mathcal{Z}$ be a maximal centered collection of zero sets in X with CIP.

Define $g(n) = \inf\{\alpha \mid L_{n,\alpha} \in \mathcal{Z}\}$. Since ordinals are well-ordered, $L_{n,g(n)}$ belongs to $\mathcal{Z}$.

In order to prove that $\bigcap \mathcal{Z} \neq \emptyset$, it suffices to show that $g \in \overline{Z}^X$ for each $Z \in \mathcal{Z}$. Let $\mathcal{U}$ be a neighborhood of g. Then there exists an $f < g$, $\mathcal{U} \supset \{y \mid f < y \leq g\}$.

The maximality of $\mathcal{Z}$ implies that $C_n = K_{n,f(n)} \cap L_{n,g(n)}$ belongs to $\mathcal{Z}$. For $Z \in \mathcal{Z}$, $Z \cap \bigcap\{C_n \mid n \in N\}$ is non-void because of CIP. But $\bigcap\{C_n \mid n \in N\} \subset \mathcal{U}$. The proof is complete.

Remark. The same method of proof can be used to show that there exists an $f \in X$ with $f \in \bigcap\{\overline{Z}^X \mid Z \in \mathcal{Z}\}$ for every maximal centered collection $\mathcal{Z}$ of zero-sets in Y with CIP.

Theorem 3. Let $P = X - \{y \in X \mid \exists n, K(y(n)) \leq \aleph_0\}$ be a subspace of X. Then $P = \upsilon Y$ (Hewitt realcompactification).

Proof. It suffices to verify the following:

I. P is realcompact,

II. Y is dense in P,

III. every point $p \in P$ is a limit of a unique maximal centered collection of zero-sets in Y with CIP.

Proof of the first part is analogous to the proof of realcompactness of X. The second statement is obvious. For the third one, consider a point $p \in P - Y$. Define $A_f^p = \{y \in Y \mid f < y \leq p\}$ for every $f < p, f \in X$ and $\mathcal{A}^p = \{A_f^p \mid f < p, f \in X\}$. Let $\mathcal{Z}$ be a maximal centered collection of zero-sets in Y, $\mathcal{Z} \supset \mathcal{A}^p$. The fact that $\mathcal{Z}$ has a countable intersection property, can be shown similarly as in Lemma 1. $\mathcal{Z}$ evidently converges to p.

$\mathcal{Z}$ is unique. Suppose the contrary: Let $\mathcal{Z}_1 \neq \mathcal{Z}$ be a maximal centered collection of zero-sets in Y with CIP which converges to p. Then there exist $Z_1 \in \mathcal{Z}_1$ and $Z \in \mathcal{Z}$, $Z_1 \cap Z = \emptyset$. Let $M = \{n \in N \mid K(p(n)) = \omega_1\}$.

For $n \in M$, let $\{s_{\alpha,n} \mid \alpha < \omega_1\}$ be an increasing sequence of ordinal numbers, converging to $p(n)$. Define an ordinal number $\beta < \omega_1$ to be odd, if there exists a limit ordinal α and a natural k such that $\beta = \alpha + 2k + 1$ and even in other cases. Since $\mathcal{Z}_1$ converges to p, $Z_1 \cap A^p_f \neq \emptyset$ for each $f < p$.

Let us choose some $f_0 \in X$, $f_0 < p$.

Let α be an ordinal, $\alpha < \omega_1$ and suppose that f_β have been defined for all $\beta < \alpha$, $f_\beta(n) < p(n)$ whenever $n \in N - M$.

a) α is odd, $\alpha = \beta + 1$. Let $g_\alpha(n) = f_\beta(n)$ for $n \in N - M$, $g_\alpha(n) = s_{\alpha,n}$ for $n \in M$. Since $Z_1 \cap A^p_g \neq \emptyset$, we can find $f_\alpha \in Z_1 \cap A^p_{g_\alpha}$ such that $f_\alpha(n) < p(n)$ for $n \in N - M$.

b) α is even, $\alpha = \beta + 1$. Similarly as in a) we shall find $f_\alpha \in Z \cap A^p_{g_\alpha}$ with $f_\alpha(n) < p(n)$ for every $n \in N - M$.

c) α is even, α is a limit ordinal. Put $h_\alpha(n) = \sup\{f_\beta(n) \mid \beta < \alpha\}$ and let $g_\alpha(n) = h_\alpha(n)$ whenever $n \in N - M$, $g_\alpha(n) = s_{\alpha,n}$ for $n \in M$. Again we shall find an $f_\alpha \in Z \cap A^p_{g_\alpha}$ with $f_\alpha(n) < p(n)$ for $n \in N - M$.

It is easy to show that the f defined by $f(n) = \sup\{f_\alpha(n) \mid \alpha < \omega_1\}$ belongs to $\overline{Z}_1 \cap \overline{Z} \cap Y = Z_1 \cap Z$, which is a contradiction.

__Theorem 4__. The space Y is closed-complete.

__Proof__. Let $\mathcal{F}$ be a maximal centered collection of closed sets in Y with CIP. Consider the system $\mathcal{Z} = \{Z \in \mathcal{F} \mid Z$

is a zero set in $Y\}$. Obviously $\mathfrak{Z}$ is a maximal centered collection of zero-sets with CIP and, if $\bigcap \mathfrak{Z} \neq \emptyset$, then $\bigcap \mathcal{F} \neq \emptyset$.

Suppose $\bigcap \mathfrak{Z} = \emptyset$. From the remark after Theorem 2 it follows that there exists some $p \in X$ with $p \in \bigcap \{\overline{Z}^X \mid Z \in \mathfrak{Z}\}$. If, for some $n \in N$, $K(p(n)) = \aleph_0$, then $\mathfrak{Z}$ has not CIP (the countable system of clopen neighborhoods of p intersected with Y belongs to $\mathfrak{Z}$ because of the maximality), which is a contradiction.

If, for each $n \in N$, there exists a k_n with $K(p(k_n)) \geq \lambda_n$, define $D'_i = \{y \in Y \mid \exists n \geq i,\ y(k_n) = p(k_n)\}$ for $i = 1, 2, 3, \ldots$. Clearly $\bigcap \{D'_i \mid i \in N\} = \emptyset$ and we may prove that $D'_i \in \mathcal{F}$ by mere modification of the proof of Theorem 1 (replace D_n by D'_n, $\mathcal{A}$ by $\mathcal{A}^p$, where $\mathcal{A}^p$ has the same meaning as in Theorem 3). We see that in this case $\mathcal{F}$ has not CIP, which is a contradiction, too.

R e f e r e n c e s

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