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CONCERNING RESOLVENT KERNELS OF VOLTERRA INTEGRAL EQUATIONS

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In this paper, a class of linear Volterra integral operators of convolution type is being investigated such that the kernel of the operator satisfies a certain linear ordinary differential equation with constant coefficients. It is shown that for every such operator there exists a linear ordinary differential equation, describing in some sense the properties of the operator. The latter differential equation makes it possible to compute effectively resolvent kernels of Volterra integral equations.

1. Notation. Let C denote the set of all complex numbers. Let R_+ denote the set of all non-negative real numbers. We shall denote by $\mathcal{C}$ the set of all continuous functions $f: R_+ \rightarrow C$ and by $\mathcal{C}^{(k)}$ (for k positive integer) the set of all k-times continuously differentiable functions $f: R_+ \rightarrow C$. Sometimes we write $\mathcal{C}^{(0)}$ instead of $\mathcal{C}$. If n, k are integers, $0 \le n \le k$, and $f \in \mathcal{C}^{(k)}$ then the symbol $f^{(n)}$ denotes the n-th derivative of the function f. Especially, $f^{(0)}$ denotes the function f itself.

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2. Definition. Let a, b be two continuous functions. A linear integral operator $T: \mathcal{C} \to \mathcal{C}$ is defined as follows:

$$Tx(t) = a(t) + \int_0^t b(t-s)\, x(s)\, ds \quad . \tag{1}$$

3. Remark. The substitution $u = t - s$ in the integral $\int_0^t b(t-s)\, x(s)\, ds$ gives

$$\int_0^t b(t-s)\, x(s)\, ds = \int_0^t x(t-u)\, b(u)\, du \tag{2}$$

and the relation (1) becomes

$$Tx(t) = a(t) + \int_0^t x(t-s)\, b(s)\, ds \quad . \tag{3}$$

4. Lemma. For any non-negative integer k and for any given functions $a, b \in \mathcal{C}^{(k)}$, the operator T maps $\mathcal{C}^{(k-1)}$ into $\mathcal{C}^{(k)}$. Moreover, for every $u \in \mathcal{C}^{(k-1)}$ and its image $v = Tu$,

$$v(t) = a(t) + \int_0^t b(t-s)\, u(s)\, ds \quad , \tag{4}$$

the following is true:

$$v^{(k)}(t) = \sum_{j=0}^{k-1} b^{(k-j-1)}(0)\, u^{(j)}(t) + a^{(k)}(t) + \int_0^t b^{(k)}(t-s)\, u(s)\, ds \quad . \tag{5.k}$$

Proof (by induction). The theorem on differentiation of an integral with respect to a parameter ensures that the function v given by (4) is differentiable if a, $b \in \mathcal{C}^{(1)}$, $u \in \mathcal{C}$. The derivative of v is then

$$v^{(1)}(t) = b(0)\,u(t) + a^{(1)}(t) + \int_0^t b^{(1)}(t-s)\,u(s)\,ds\,.$$

Thus the operator T maps $\mathscr{C}^{(0)}$ into $\mathscr{C}^{(1)}$; hence (5.1) holds. Now, let us suppose that

$$(5.k-1)\qquad v^{(k-1)}(t) = \sum_{j=0}^{k-2} b^{(k-j-2)}(0)\,u^{(j)}(t) + $$
$$+ a^{(k-1)}(t) + \int_0^t b^{(k-1)}(t-s)\,u(s)\,ds$$

holds and furthermore $a, b \in \mathscr{C}^{(k)}$, $u \in \mathscr{C}^{(k-1)}$. Then the function $v^{(k-1)}$ is continuously differentiable and the differentiation with respect to t on both sides of (5.k-1) gives

$$v^{(k)}(t) = \sum_{j=0}^{k-2} b^{(k-j-2)}(0)\,u^{(j+1)}(t) + b^{(k-1)}(0)\,u(t) +$$
$$+ a^{(k)}(t) + \int_0^t b^{(k)}(t-s)\,u(s)\,ds\,.$$

Substituting $j+1 \longrightarrow j$ in the last equation (5.k) is easily obtained.

5. <u>Remark</u>. From (5.k) and (2) it follows immediately

$$(6.k)\qquad v^{(k)}(t) = \sum_{j=0}^{k-1} b^{(k-j-1)}(0)\,u^{(j)}(t) + a^{(k)}(t) +$$
$$+ \int_0^t u(t-s)\,b^{(k)}(s)\,ds\,.$$

Supposing now $a, u \in \mathscr{C}^{(k)}$, $b \in \mathscr{C}^{(k-1)}$, and using (2) to modify (4) to the form

$$(7)\qquad v(t) = a(t) + \int_0^t u(t-s)\,b(s)\,ds\,,$$

we obtain from Lemma 4:

$$(8.k) \qquad v^{(k)}(t) = \sum_{j=0}^{k-1} u^{(k-j-1)}(0)\, b^{(j)}(t) + a^{(k)}(t) + \\ + \int_0^t u^{(k)}(t-s)\, b(s)\, ds .$$

This may also be written, using the relation (2), as

$$(9.k) \qquad v^{(k)}(t) = \sum_{j=0}^{k-1} u^{(k-j-1)}(0)\, b^{(j)}(t) + a^{(k)}(t) + \\ + \int_0^t b(t-s)\, u^{(k)}(s)\, ds .$$

Relations (5.k),(6.k),(8.k) and (9.k) make it possible for the operator T to be conveniently characterized by certain linear differential operators.

6. __Theorem__. Let $A_0, A_1, \dots, A_m$ be complex constants and $a, b \in \mathcal{C}^{(m)}$. Let b be the solution of the initial value problem

$$(10) \sum_{k=0}^{m} A_k x^{(k)} = 0 ,\ x^{(k)}(0) = b^{(k)}(0) = b_k ,\ k = 0,1,\dots,m-1 .$$

If $u \in \mathcal{C}^{(m-1)}$, then the function $v = Tu$ is the solution of the initial value problem

$$(11) \qquad \sum_{k=0}^{m} A_k y^{(k)} = \sum_{k=0}^{m-1} B_k u^{(k)}(t) + \sum_{k=0}^{m} A_k a^{(k)}(t) ,$$

where

$$B_k = \sum_{j=k+1}^{m} A_j b_{j-k-1}$$

with the initial conditions

$$(12) \quad v(0) = a(0),\ v^{(j)}(0) = b_{j-1} u(0) + b_{j-2} u^{(1)}(0) + \dots + \\ + b_0 u^{(j-1)}(0) + a^{(j)}(0) , \quad j = 1, 2, \dots, m-1 .$$

Proof. Let

$$v(t) = a(t) + \int_0^t b(t-s)\,u(s)\,ds .$$

Then, according to (6.k), there holds for all integers $0 < k \le n$

$$A_k v^{(k)}(t) = A_k \sum_{j=0}^{k-1} b_{k-j-1} u^{(j)}(t) + A_k a^{(k)}(t) + \int_0^t u(t-s) A_k b^{(k)}(s)\,ds .$$

Hence, summing over all k's from 0 to n and using the assumption $\sum_{k=0}^{n} A_k b^{(k)}(s) = 0$, we have

$$\sum_{k=0}^{n} A_k v^{(k)}(t) = \sum_{k=0}^{n} A_k \sum_{j=0}^{k-1} b_{k-j-1} u^{(j)}(t) + \sum_{k=0}^{n} A_k a^{(k)}(t) .$$

From the Dirichlet's formula for double sums we obtain

$$\sum_{k=0}^{n} A_k v^{(k)}(t) = \sum_{j=0}^{n-1} u^{(j)}(t) \sum_{k=j+1}^{n} A_k b_{k-j-1} + \sum_{k=0}^{n} A_k a^{(k)}(t) ,$$

which is equivalent to (11).

7. Remark. If the function a also satisfies (10), then (11) becomes

$$(13) \quad \sum_{k=0}^{n} A_k y^{(k)} = \sum_{k=0}^{n-1} B_k u^{(k)}(t); \quad B_k = \sum_{j=k+1}^{n} A_j b_{j-k-1} .$$

Let Theorem 6 be illustrated by two simple examples.

8. Examples. 1. Let $a \in \mathcal{C}^{(2)}$ be arbitrary and

$$b(t) = \beta_1 e^{\lambda_1 t} + \beta_2 e^{\lambda_2 t}, \quad \beta_1, \beta_2, \lambda_1, \lambda_2 \in C .$$

Then $b_0 = \beta_1 + \beta_2$, $b_1 = \lambda_1 \beta_1 + \lambda_2 \beta_2$, $A_0 = \lambda_1 \lambda_2$,

$A_1 = -(\lambda_1 + \lambda_2)$, $A_2 = 1$, $B_0 = -(\beta_1 \lambda_2 + \beta_2 \lambda_1)$, $B_1 = \beta_1 + \beta_2$.

For any $x \in \mathcal{C}^{(1)}$ the function

$$y(t) = a(t) + \int_0^t (\beta_1 e^{\lambda_1 (t-s)} + \beta_2 e^{\lambda_2 (t-s)})\, x(s)\,ds$$

solves the following initial value problem

$$y^{(2)} - (\lambda_1 + \lambda_2) y^{(1)} + \lambda_1 \lambda_2 y = \lambda_1 \lambda_2 a(t) - (\lambda_1 + \lambda_2) a^{(1)}(t) +$$

$+ a^{(2)}(t) - (\beta_1 \lambda_2 + \beta_2 \lambda_1) x(t) + (\beta_1 + \beta_2) x^{(1)}(t)$,
with the initial conditions

$$y(0) = a(0), \; y^{(1)}(0) = a^{(1)}(0) + (\beta_1 + \beta_2) x(0) .$$

2. Let $a(t) = \sum_{i=0}^{m-1} a_i t^i$ be an arbitrary polynomial and $b(t) = t^{m-1}$.

Then $b_j = 0$ for $j = 0, 1, \dots, m-2$, $b_{m-1} = (m-1)!$, $A_j = 0$ for $j = 0, 1, \dots, m-1$, $A_m = 1$, $B_0 = (m-1)!$, $B_j = 0$ for $j = 1, 2, \dots, m-1$.

For any $x \in \mathcal{C}^{(m-1)}$ the function

$$y(t) = a(t) + \int_0^t (t-s)^{m-1} x(s)\, ds$$

solves the following initial value problem

$$y^{(m)} = (m-1)!\, x(t), \; y^{(j)}(0) = a^{(j)}(0) = j!\, a_j, \; j = 0, 1, \dots, m-1 .$$

9. <u>Remark</u>. Theorem 6 may serve as a useful tool for the computation of fixed points of the integral operator (1) or, which amounts to the same, for the solution of a Volterra integral equation of the second kind. Actually, a function $x \in \mathcal{C}^{(m)}$ is a fixed point of the operator T iff x is the solution of the equation

$$x(t) = a(t) + \int_0^t b(t-s)\, x(s)\, ds, \quad t \geq 0 .$$

It then follows from Theorem 6 that x solves the initial value problem

$$\sum_{k=0}^{m} C_k x^{(k)} = \sum_{k=0}^{m} A_k a^{(k)}(t), \tag{14}$$

$$C_k = A_k - \sum_{j=k+1}^{m} A_j b_{j-k-1}, \; k = 0, 1, \dots, m-1, \; C_m = A_m ,$$

with the initial conditions

$$x(0) = a(0), \; x^{(j)}(0) = \sum_{k=0}^{j-1} b_{j-k-1} x^{(k)}(0) + a^{(j)}(0), \tag{15}$$

$$j = 1, 2, \dots, m-1 .$$

The two examples in 8 show that the solution of

$$x(t) = a(t) + \int_0^t (\beta_1 e^{\lambda_1(t-s)} + \beta_2 e^{\lambda_2(t-s)})\, x(s)\, ds$$

may be found by solving the initial value problem

$$x^{(2)} - (\lambda_1 + \lambda_2 + \beta_1 + \beta_2)\, x^{(1)} + (\lambda_1 \lambda_2 + \beta_1 \lambda_2 + \beta_2 \lambda_1)\, x =$$

$$= \lambda_1 \lambda_2\, a(t) - (\lambda_1 + \lambda_2)\, a^{(1)}(t) + a^{(2)}(t) ,$$

$$x(0) = a(0),\ x^{(1)}(0) = (\beta_1 + \beta_2)\, a(0) + a^{(1)}(0) .$$

Similarly, the solution of the integral equation

$$x(t) = \sum_{i=0}^{m-1} a_i t^i + \int_0^t (t-s)^{m-1} x(s)\, ds$$

may be obtained by solving the initial value problem

$$x^{(m)} - (m-1)!\, x = 0,\ x^{(j)}(0) = j!\, a_j ,\ j = 0, 1, \dots, m-1 .$$

Since the kernels of the type $k(t) = \sum_{i=1}^{m} \beta_i e^{\lambda_i t}$ occur quite frequently in many practical problems of the control theory, an explicit formula for the corresponding initial value problem is given below.

10. Example. The solution of the integral equation

$$x(t) = a(t) + \int_0^t (\sum_{i=1}^{m} \beta_i e^{\lambda_i(t-s)})\, x(s)\, ds$$

may be found by solving the initial value problem (14), (15). The numbers A_k in (14) are now the coefficients of the polynomial

$$P(\lambda) = \sum_{k=0}^{m} A_k \lambda^k = \prod_{i=1}^{m} (\lambda - \lambda_i) .$$

It is known that the coefficients A_k may be expressed by the roots λ_i as follows:

$$A_m = 1 ,$$

$$A_{m-k} = (-1)^k \sum_{\substack{i_1, i_2, \dots, i_k \\ i_1 < i_2 < \dots < i_k}} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_k} ,\ k = 1, 2, \dots, m .$$

The numbers b_k, $a_k = a^{(k)}(0)$ in (15) are given by the relation

$$b_k = \sum_{i=1}^{m} \beta_i \lambda_i^k , \qquad k = 0, 1, \dots, m-1 .$$

11. Remark. In the analysis of a linear integral equation

(16) $$x(t) = a(t) + \int_0^t b(t-s)\, x(s)\, ds$$

a very important role is played by its resolvent kernel r given as a solution of a linear integral equation

(17) $$r(t) = b(t) + \int_0^t b(t-s)\, r(s)\, ds .$$

It is well-known that if a function r is a resolvent kernel of Equation (16), then the solution x of (16) may be expressed as

(18) $$x(t) = a(t) + \int_0^t r(t-s)\, a(s)\, ds .$$

Since the resolvent equation (17) is again a linear Volterra integral equation, Theorem 6 or its modifications in Remarks 7 and 9 may be applied. Thus the following theorem may be formulated:

12. Theorem. Let the function $b \in \mathcal{C}^{(m)}$ be a solution of the equation $\sum_{k=0}^{m} A_k x^{(k)} = 0$. Then the resolvent kernel r of the equation (16) satisfies the initial value problem

(19) $$\sum_{k=0}^{m} C_k x^{(k)} = 0 , \quad C_k = A_k - \sum_{j=k+1}^{m} A_j b_{j-k-1} ,$$
$$j = 0, 1, \dots, m-1 , \quad C_m = A_m ,$$

with the initial conditions

$$(20) \qquad x^{(j)}(0) = \sum_{k=0}^{j} b_{j-k}\, r^{(k-1)}(0), \quad j = 0, 1, \dots, m-1,$$

where $r^{(-1)}(0) = 1$.

13. Remark. It may be seen that finding the solution of the initial value problem (19),(20) for the resolvent kernel is easier than solving the initial value problem for the solution of Equation (16) itself. Moreover, if the resolvent kernel r of Equation (16) is known, any solution of Equation (16) with an arbitrary right hand side $a(t)$ is found by integration using Relation (18). On the other hand, when using Equation (14), the corresponding particular integral of this equation has to be computed for each particular choice of the function $a(t)$.

Now, let us apply Theorem 12 to find the resolvent kernel of the integral equation from Example 10.

14. Examples. 1. The resolvent kernel of the linear Volterra integral equation

$$(21) \qquad x(t) = a(t) + \int_0^t \Big(\sum_{i=1}^{m} \beta_i\, e^{\lambda_i (t-s)}\Big)\, x(s)\, ds$$

satisfies the initial value problem (19),(20) with the coefficients A_k, b_k described in Example 10. In a special case, e.g. for $m = 2$, the initial value problem has the form

$$(22) \qquad x^{(2)} - (\lambda_1 + \lambda_2 + \beta_1 + \beta_2)\, x^{(1)} + (\lambda_1 \lambda_2 + \beta_1 \lambda_2 + \beta_2 \lambda_1)\, x = 0,$$

$$x(0) = \beta_1 + \beta_2, \quad x^{(1)}(0) = (\beta_1 + \beta_2)^2 + \beta_1 \lambda_1 + \beta_2 \lambda_2$$

Let μ_1, μ_2 be characteristic roots of Equation (22), $\mu_1 \neq \mu_2$. The resolvent kernel r is then

$$r(t) = K_1 e^{\mu_1 t} + K_2 e^{\mu_2 t}$$

with

$$K_1 = \frac{1}{\mu_1 - \mu_2} [(\beta_1+\beta_2)^2 + \beta_1\lambda_1 + \beta_2\lambda_2 - (\beta_1+\beta_2)\mu_2] ,$$

$$K_2 = \frac{1}{\mu_1 - \mu_2} [(\beta_1+\beta_2)\mu_1 - (\beta_1+\beta_2)^2 - \beta_1\lambda_1 - \beta_2\lambda_2] .$$

In the special case $b(t) = 1 - e^{-t}$ which often occurs e.g. in the theory of phase controlled oscillations, the resolvent kernel is obtained as the solution of the initial value problem

$$x^{(2)} + x^{(1)} - x = 0, \quad x(0) = 0 , \quad x^{(1)}(0) = 1 .$$

Setting $\mu_1 = \frac{-1+\sqrt{5}}{2}$, $\mu_2 = \frac{-1-\sqrt{5}}{2}$ the resolvent kernel r of the integral equation with the kernel $b(t) = 1 - e^{-t}$ has the form

$$r(t) = \frac{1}{\sqrt{5}} (e^{\mu_1 t} - e^{\mu_2 t}) .$$

2. The resolvent kernel r for an integral equation with the kernel $b(t) = \sum_{i=0}^{m} a_i t^i$ may be found as the solution of the initial value problem (19),(20) as follows. The polynomial b is the solution of the differential equation $x^{(m+1)} = 0$. Thus $A_{m+1} = 1$, $A_k = 0$ for $k = 0, 1, \ldots, m$, $b_k = k!\, a_k$ for $k = 0, 1, \ldots, m$.

Hence, for the coefficients C_k of Equation (19) we obtain

$$C_0 = 1, \quad C_k = A_{m-k+1} - \sum_{j=1}^{k} A_{m+1-k+j}\, b_{j-1} = -b_{k-1} = -(k-1)!\, a_{k-1}$$

for $k = 1, 2, \ldots, m+1$.

Thus the resolvent kernel solves the initial value problem

$$x^{(n+1)} - a_0 x^{(n)} - a_1 x^{(n-1)} - \dots - (n-1)!\, a_{n-1} x^{(1)} - n!\, a_n x = \\ = 0 \quad ,$$

$$x^{(j)}(0) = \sum_{k=0}^{j} (j-k)!\, a_{j-k}\, x^{(k-1)}(0), \quad j = 0, 1, 2, \dots, n \ .$$

3. A procedure similar to that described above leads to differential equations for resolvent kernels of integral equations having kernels of the type $b(t) = P(t)\, e^{\lambda t}$ with $P(t)$ a polynomial of the degree $n-1$. It is obvious that the function b is a solution of an ordinary linear differential equation of the order n with constant coefficients

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \lambda^k x^{(n-k)} = 0 \ ,$$

with λ the characteristic root of multiplicity n . Hence

$$A_{n-k} = (-1)^k \binom{n}{k} \lambda^k, \quad b_k = b^{(k)}(0) = \sum_{j=0}^{k} \binom{k}{j} P^{(k-j)}(0)\, \lambda^j \ ,$$

$$C_i = (-1)^i \binom{n}{i} \lambda^i - \\ - \sum_{k=1}^{i} (-1)^{i-k} \binom{n}{i-k} \lambda^{i-k} \sum_{j=0}^{k-1} \binom{k-1}{j} P^{(k-j-1)}(0)\, \lambda^j \ .$$

Now, substituting these values of the constants C_k , b_k into (19) and (20), the initial value problem for the resolvent kernel is obtained.

15. <u>Remark</u>. The method described above leads to algebraic equations, whose roots will eventually have to be computed. Here we meet with the same difficulty as when using the Laplace transforms. Notwithstanding, in several special cases our method is more convenient and the pro-

cess of computing the resolvent kernels is very simple. The procedure just described may be modified in many cases in a variety of ways depending on the special form of the kernel. One of such modifications will be shown in what follows.

16. <u>Example</u>. Take the same kernel as in Example 14.3, that is, $k(t) = P(t)e^{\lambda t}$ and the resolvent equation in the form

$$(22)\qquad r(t) = P(t)e^{\lambda t} + \varepsilon \int_0^t r(t-s)P(s)e^{\lambda s}\,ds$$

where ε may be either 1 or -1. Setting $v(t) = k(t) = \varepsilon r(t)$, $u(t) = a(t) = P(t)e^{\lambda t}$ into (5.k) we obtain

$$r^{(k)}(t) = \varepsilon \sum_{j=0}^{k-1} r^{(k-j-1)}(0)(P(t)e^{\lambda t})^{(j)} + (P(t)e^{\lambda t})^{(k)} + \varepsilon \int_0^t r^{(k)}(t-s)P(s)e^{\lambda s}\,ds.$$

Denoting $r^{(-1)}(0) = \varepsilon$ and introducing the abbreviated notation $r^{(j)}(0) = r_j$ for $j = -1, 0, 1, 2, \ldots$;

$$H_k(t) = \varepsilon \sum_{j=0}^{k} r^{(k-j-1)}(0)(P(t)e^{\lambda t})^{(j)} ,$$

we have

$$\begin{aligned} H_k(t) &= \varepsilon \sum_{j=0}^{k} r_{k-j-1} \sum_{q=0}^{j} \binom{j}{q} P^{(q)}(t)\lambda^{j-q}e^{\lambda t} = \\ &= \varepsilon \sum_{q=0}^{k} P^{(q)}(t) \sum_{j=q}^{k} r_{k-j-1} \binom{j}{q} \lambda^{j-q} e^{\lambda t} = \\ &= \varepsilon \sum_{q=0}^{k} \frac{P^{(q)}(t)}{q!} \sum_{j=q}^{k} j(j-1)\ldots(j-q+1) r_{k-j-1}\lambda^{j-q}e^{\lambda t} = \\ &= \varepsilon \sum_{q=0}^{k} \frac{P^{(q)}(t)}{q!} Q_k^{(q)}(\lambda)e^{\lambda t} , \end{aligned}$$

where

$$(23)\quad Q_k(\lambda) = Q_k^{(0)}(\lambda) = \sum_{j=0}^{k} \kappa_{k-j-1} \lambda^j , \; Q_k^{(q)}(\lambda) = \sum_{j=q}^{k} j(j-1) \cdot \ldots \cdot (j-q+1) \kappa_{k-j-1} \lambda^{j-q} .$$

Clearly, the polynomial $Q_k^{(q)}(\lambda)$ is the q-th derivative of the polynomial $Q_k(\lambda)$. Thus the k-th derivative of both the sides of the resolvent equation (22) may be written in the form

$$(24.k)\quad \kappa^{(k)}(t) = \varepsilon \sum_{q=0}^{k} \frac{P^{(q)}(t)\, Q_k^{(q)}(\lambda)}{q!} e^{\lambda t} + \varepsilon \int_0^t \kappa^{(k)}(t-s) P(s) e^{\lambda s} \, ds .$$

17. Remark. Equation (24.k) appears to be a very useful tool when investigating the various qualitative and quantitative properties of derivatives of the resolvent kernels of Volterra integral equations having kernels of the form $P(t)e^{\lambda t}$. One illustration of such applications is given in the next example.

18. Example. Let us investigate the following problem. Does there exist a polynomial $P(t) = a_0 + a_1 t + \ldots + a_{k-1} t^{k-1}$ of the degree $k-1$ such that the resolvent kernel κ, corresponding to the Volterra kernel $b(t) = P(t) e^{\lambda t}$, will also be a polynomial of the same degree? We shall find the conditions of the existence of such polynomial. Since κ is required to be a polynomial of the degree $k-1$, its k-th derivative has to be identically zero. Since the function $\kappa^{(k)}$ is a solution of Equation (24.k), the following must hold:

$$\varepsilon \sum_{q=0}^{k} \frac{P^{(q)}(t)\, Q_k^{(q)}(\lambda)}{q!} e^{\lambda t} \equiv 0 .$$

Hence all coefficients $Q_k^{(q)}(\lambda)$ for $q = 0,1,\dots,k-1$ have to be zero. Thus λ is the root of multiplicity k of the polynomial $Q_k(\lambda)$. Consequently, $Q_k(x) = \varepsilon.(x-\lambda)^k$. Comparing this with (23) we obtain

$$Q_k(x) = \varepsilon(x-\lambda)^k = \varepsilon \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} x^j \lambda^{k-j} = \sum_{j=0}^{k} \kappa_{k-j-1} x^j ,$$

so that

$$\kappa_{k-j-1} = \varepsilon(-1)^{k-j} \binom{k}{j} \lambda^{k-j}, \quad j = 0,1,\dots,k .$$

Setting $i = k - j - 1$ we have

$$\kappa_i = (-1)^{i+1} \varepsilon \binom{k}{i+1} \lambda^{i+1} .$$

Since the number $\dfrac{\kappa_i}{i!}$ is the i-th coefficient of the resolvent kernel κ, we have finally obtained an explicit formula for the resolvent kernel

$$\kappa(t) = \sum_{i=0}^{k-1} (-1)^{i+1} \varepsilon \binom{k}{i+1} \frac{\lambda^{i+1}}{i!} t^i .$$

Now, the coefficients $a_i = \dfrac{P^{(i)}(0)}{i!}$ of the polynomial $P(t)$ remain to be found. Equation (22) may be rewritten in the form

$$P(t) = \kappa(t) e^{-\lambda t} - \varepsilon \int_0^t \kappa(t-s) e^{-\lambda(t-s)} P(s)\, ds .$$

According to (5.k), the k-th derivative of $P(t)$ is

$$P^{(k)}(t) = -\varepsilon \sum_{j=0}^{k-1} P^{(k-j-1)}(0) \left(\kappa(t) e^{-\lambda t}\right)^{(j)} + \left(\kappa(t) e^{-\lambda t}\right)^{(k)} -$$
$$- \varepsilon \int_0^t \kappa(t-s) e^{-\lambda(t-s)} P^{(k)}(s)\, ds .$$

Setting

P_{k-j-1} for $P^{(k-j-1)}(0)$, $j = 0,1,\dots,k-1$, $P^{(-1)}(0) = P_{-1} = -\varepsilon$, we obtain

$$\widetilde{H}_k(t,\lambda) = -\varepsilon \sum_{j=0}^{k} P_{k-j-1} \left(\varkappa(t)e^{-\lambda t}\right)^{(j)} =$$

$$= -\varepsilon \sum_{j=0}^{k} P_{k-j-1} \sum_{q=0}^{j} \binom{j}{q} \quad \varkappa^{(q)}(t)(-\lambda)^{j-q} e^{-\lambda t} =$$

$$= -\varepsilon \sum_{q=0}^{k} \frac{\varkappa^{(q)}(t)}{q!} \sum_{j=q}^{k} j(j-1)\cdot\ldots\cdot(j-q+1) P_{k-j-1}(-\lambda)^{j-q} e^{-\lambda t} =$$

$$= -\varepsilon \sum_{q=0}^{k} \frac{\varkappa^{(q)}(t)}{q!} \widetilde{Q}_k^{(q)}(-\lambda)e^{-\lambda t} \quad ,$$

where

$$(25) \qquad \widetilde{Q}_k(-\lambda) = \widetilde{Q}_k^{(0)}(-\lambda) = \sum_{j=0}^{k} (-1)^j P_{k-j-1} \lambda^j \quad ,$$

and

$$\widetilde{Q}_k^{(q)}(-\varkappa) = \sum_{j=q}^{k} j(j-1)\cdot\ldots\cdot(j-q+1) P_{k-j-1}(-\lambda)^{j-q} \; ,$$

which is the q-th derivative of $\widetilde{Q}_k(-\lambda)$.

Since $P(t)$ is a polynomial of the degree $k-1$, necessarily $P^{(k)}(t) = 0$ for all t , and hence, similarly as above, $\widetilde{Q}_k^{(q)}(-\lambda) = 0$ for all $q = 0, 1, \ldots$ $\ldots, k-1$. Thus $-\lambda$ is the root of multiplicity k of the polynomial $\widetilde{Q}_k(-\lambda)$.

Hence and from (25) we have

$$\widetilde{Q}_k^{(x)} = P_{-1}(x+\lambda)^k =$$

$$= \sum_{j=0}^{k} (-\varepsilon)\binom{k}{j} \lambda^{k-j} x^j = \sum_{j=0}^{k} P_{k-j-1} x^j \quad ,$$

and thus

$$P_{k-j-1} = -\varepsilon \binom{k}{j} \lambda^{k-j} \; .$$

Setting $i = k - j - 1$, we obtain

$$a_i = \frac{P_i}{i!} = -\varepsilon \binom{k}{i+1} \frac{\lambda^{i+1}}{i!} \quad .$$

In this way, the following result has been obtained: for each kernel of the form $k(t) = P(t)e^{\lambda t}$ with $P(t) = -\varepsilon \sum_{i=0}^{k-1} \binom{k}{i+1} \frac{\lambda^{i+1}}{i!} t^i$ the corresponding resolvent kernel $\varkappa$ is the polynomial

$$\varkappa(t) = -\varepsilon \sum_{i=0}^{k-1} (-1)^i \binom{k}{i+1} \frac{\lambda^{i+1}}{i!} t^i \quad .$$

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ON THE CONVERGENCE OF SEQUENCES OF LINEAR OPERATORS AND ADJOINT OPERATORS

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1. Introduction

Let X and Y be two Banach spaces with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. X^* (resp. Y^*) denotes the adjoint space of all bounded linear functionals on X (resp. on Y). The pairing between $x^* \in X^*$ and $x \in X$ is denoted by $\langle x, x^* \rangle_X$ (analogously for $y^* \in Y^*$ and $y \in Y$). We shall use the symbols $\xrightarrow{X}$, $\xrightarrow{X}$ to denote the strong convergence in X and the weak convergence in X, respectively. If $\mathcal{L}(X, Y)$ is the space of all bounded linear operators from X into Y then the convergence of a sequence $(A_n) \subset \mathcal{L}(X, Y)$ can be considered in various meaning. We shall consider the following types.

Definition 1. Let $A \in \mathcal{L}(X,Y)$, $(A_n) \subset \mathcal{L}(X,Y)$. Then

(i) (A_n) is said to be converged to A if $A_n x \xrightarrow{Y} Ax$ for any $x \in X$.

(ii) (A_n) is said to be continuously converged to A if $A_n x_n \xrightarrow{Y} Ax$ for any $(x_n) \subset X$, $x_n \xrightarrow{X} x$.

(iii) $(A_n$ is said to be weakly converged to A if

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$A_n x \xrightarrow{Y} Ax$ for any $x \in X$.

(iv) (A_n) is said to be weakly continuously converged to A if $A_n x_n \xrightarrow{Y} Ax$ for any $(x_n) \subset X$, $x_n \xrightarrow{X} x$.

The convergence of (A_n) to A in the meaning of (i) or (ii),(iii),(iv) is denoted by $A_n \longrightarrow A$ or $A_n \xrightarrow{c} A$, $A_n \longrightarrow A$, $A_n \xrightarrow{c} A$, respectively.

The relations among these types of the convergence are examined in Section 2.

Let A^* denote the adjoint operator to $A \in \mathcal{L}(X,Y)$, i.e. A^* is such an element of $\mathcal{L}(Y^*, X^*)$ that $\langle Ax, y^* \rangle_Y = \langle x, A^* y^* \rangle_X$ for any $x \in X$, $y^* \in Y^*$. It can be shown that $A_n \longrightarrow A$ does not imply $A_n^* \longrightarrow A^*$ (see Example 1 in Section 2 or Yosida [4], Chap.VII,§ 1, Prop.1). In Proposition 2 and Theorem 1 we shall give the sufficient and necessary condition under that $A_n^* \longrightarrow A^*$. The special case of operators with norms equal to 1 is given in Theorem 2 and in its Corollary. Solving the problem when the convergence (i) implies the convergence (iv) for any sequence $(A_n) \subset \mathcal{L}(X,Y)$, we obtain a new characterization of Banach spaces with finite dimension (Theorem 3). The convergence of adjoint operators is important for instance in the case that $X = Y$ and (A_n) are projections (i.e. $A_n^2 = A_n$), $A = I$ (I denotes the identity operator) - see e.g. Browder [1]). Except rewriting the main results of Sections 2 we shall give the conditions for the convergence of adjoint projections in the sense of (i) in Definition 1,in Section 3.

2. The relations among various types of the convergence

Two relations are obvious, namely $A_n \longrightarrow A$ or $A_n \xrightarrow{c} A$ implies $A_n \rightharpoonup A$.

Proposition 1. $A_n \longrightarrow A$ if and only if $A_n \xrightarrow{c} A$.

Proof. As from $A_n \xrightarrow{c} A$ it obviously follows that $A_n \rightharpoonup A$, we have only to prove the necessary part. If $A_n \rightharpoonup A$, then, by virtue of the Banach-Steinhaus theorem (see e.g. Yosida [4]), there exists a positive number K such that $\| A_n \| \leq K$ for any positive integer n . Let now $x_n \xrightarrow{X} x$. By the triangle inequality, we have
$\| A_n x_n - Ax \|_Y \leq K \| x_n - x \|_X + \| A_n x - Ax \|_Y$.
It follows that $A_n \xrightarrow{c} A$.

An analogous statement for the weak convergence does not hold as it will be shown in the sequel. The following two statements make clear the notion of weakly continuously converging sequences.

Proposition 2. If $A_n^* \longrightarrow A^*$ then $A_n \xrightarrow{c} A$.

Proof. Let (x_n) be such a sequence of elements of X that $x_n \xrightarrow{X} x$ and let $y^* \in Y^*$. Then
$\langle A_n x_n, y^* \rangle_Y = \langle x_n, A_n^* y^* \rangle_X \rightarrow \langle x, A^* y^* \rangle_X = \langle Ax, y^* \rangle_Y$
because $A_n^* y^* \xrightarrow{X^*} A^* y^*$. Therefore $A_n \xrightarrow{c} A$.

Theorem 1. Let X be a separable and reflexive Banach space and let Y be a Banach space. Then from $A_n \xrightarrow{c} A$ it follows that $A_n^* \longrightarrow A^*$.

Proof. According to Kadec [3] there exists a norm $||| \cdot |||_{X^*}$ which is equivalent to the norm $\| \cdot \|_{X^*}$ generated by the norm $\| \cdot \|_X$ in X and which has the following property

(P) If $x_n^* \xrightarrow{X^*} x^*$, $|||x_n^*|||_{X^*} \longrightarrow |||x^*|||_{X^*}$

then $x_n^* \xrightarrow{X^*} x^*$.

The norm $|||\cdot|||_{X^*}$ on X generates the new norm $|||\cdot|||_X$ on X by the relation

$$|||x|||_X = \sup_{|||x^*|||_{X^*} \le 1} |\langle x, x^* \rangle_X| \qquad \text{for } x \in X .$$

The norm $|||\cdot|||_X$ on X is also equivalent to the previous norm $\|\cdot\|_X$. Let now $A_n \xrightarrow{c} A$. Then $A_n \longrightarrow A$ and therefore, by using the reflexivity of X , also $A_n^* \longrightarrow A^*$. If $y^* \in Y^*$ then

$$|||A^* y^*|||_{X^*} \le \liminf_{n \to \infty} |||A_n^* y^*|||_{X^*} .$$

By virtue of the Hahn-Banach theorem, there exists a sequence (x_n) of elements of the sphere $S = \{x \in X , |||x|||_X = 1\}$ such that

$$|||A_n^* y^*|||_{X^*} = \langle x_n, A_n^* y^* \rangle_X = \langle A_n x_n, y^* \rangle_Y .$$

As X is reflexive, the sphere S is relatively weakly sequentially compact (see e.g. Day [2]) and therefore there exists a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \xrightarrow{X} x \in X$. Putting $z_{n_k} = x_{n_k}$ and $z_n = x$ for $n \neq n_k$ we have

$$|||A^* y^*|||_{X^*} \le \liminf_{n \to \infty} |||A_n^* y^*|||_{X^*} =$$

$$= \liminf_{n \to \infty} \langle A_n x_n, y^* \rangle_Y \le$$

$$\leq \liminf_{k\to\infty} \langle A_{n_k} x_{n_k}, y^* \rangle_Y = \lim_{n\to\infty} \langle A_n x_n, y^* \rangle_Y =$$

$$= \langle Ax, y^* \rangle_Y = \langle x, A^* y^* \rangle_X \leq$$

$$\leq |||x|||_X \cdot |||A^* y^*|||_{X^*} \leq |||A^* y^*|||_{X^*} .$$

Hence

$$|||A^* y^*|||_{X^*} = \liminf_{n\to\infty} |||A_n^* y^*|||_{X^*} .$$

We shall now prove that also

$$|||A^* y^*|||_{X^*} = \limsup_{n\to\infty} |||A_n^* y^*|||_{X^*} .$$

If it is not this case, i.e. $\limsup_{n\to\infty} |||A_n^* y^*|||_{X^*} >$ $> |||A^* y^*|||_{X^*}$ then there exists such a subsequence (n_k) of positive integers that

$$|||A^* y^*|||_{X^*} < \lim_{k\to\infty} |||A_{n_k}^* y^*|||_{X^*} .$$

By the same manner as above we get the contradiction.

Summarizing, we have $A_n^* y^* \xrightarrow{X^*} A^* y^*$ and $|||A_n^* y^*|||_{X^*} \longrightarrow |||A^* y^*|||_{X^*}$ and thus, by the validity of Property (P), we obtain that

$|||A_n^* y^* - A^* y^*|||_{X^*} \longrightarrow 0$. By using the equivalence property of the norms $|||\cdot|||_{X^*}$, $\|\cdot\|_{X^*}$, it is $A_n^* \to$ $\longrightarrow A^*$ which was to be proved.

<u>Remark 1</u>. We have heard that S.L. Trojanski proved that the Kadec theorem takes place in the case that X is reflexive and not necessarily separable (to appear in Studia Mathematica, vol.37). Therefore, the assumption of separability can be omitted in Theorem 1. We shall use this remark in the sequel.

Corollary 1. Let X and Y be two Banach spaces and let X be a reflexive space. Then the following conditions are equivalent:

(i) $A_n^* \longrightarrow A^*$.

(ii) $A_n^* \xrightarrow{c} A^*$.

(iii) $A_n \xrightarrow{c} A$.

(iv) $(\|A_n\|)$ is a bounded sequence and $A_n^* y^* \xrightarrow{X^*} A^* y^*$ for $y^* \in \mathfrak{D}^*$ where the linear hull of $\mathfrak{D}^*$ is dense in Y^*.

Proof. The equivalence of (i) and (ii) is stated in Proposition 1, the equivalence of (i) and (iii) is proved in Proposition 2, Theorem 1 and Remark 1, the equivalence of (i) and (iv) is the Banach-Steinhaus theorem. .

Corollary 2. Let X and Y be two Banach spaces and let Y be a reflexive space. Then the following conditions are equivalent:

(i) $A_n \longrightarrow A$.

(ii) $A_n \xrightarrow{c} A$.

(iii) $A_n^* \xrightarrow{c} A^*$.

(iv) $(\|A_n\|)$ is a bounded sequence and $A_n x \xrightarrow{Y} Ax$ for $x \in \mathfrak{D}$, where the linear hull of $\mathfrak{D}$ is dense in X .

The following example shows that there is no relation between $A_n \longrightarrow A$ and $A_n^* \longrightarrow A^*$ even in the case of projections on Hilbert spaces.

Example 1. Let $X = \ell^2$, $e_n = (\delta_{in})_i$ (δ_{in} is the Kronecker symbol) and P_n be the orthogonal projection onto $X_n = Lin(e_1,\dots,e_n)$ (Lin stands here for the linear hull). For $x = \sum_{i=1}^{+\infty} \xi_i e_i \in \ell^2$ we put $T_n x = \xi_{n+1} e_1 + \dots + \xi_{2n} e_n$ and $Q_n = P_n + T_n$. Then $Q_n^2 = Q_n$, i.e. Q_n is a projection onto X_n, $Q_n \longrightarrow I$ (I denotes the identity operator). From $e_n \xrightarrow{X} \theta$ and $Q_n e_{n+1} = e_1$ we see that $Q_n \overset{c}{\not\longrightarrow} I$ and, by virtue of Proposition 2, the sequence (Q_n^*) does not converge to I^*. One can easily show that $Q_n^* x \longrightarrow x$ if and only if $x = \theta$.

Remark 2. Example 1 shows that $A_n \longrightarrow A$ does not imply $A_n \xrightarrow{c} A$. By the same manner ($Q_n^* \xrightarrow{c} I^*$ as it follows from Corollary 2) $A_n \xrightarrow{c} A$ does not imply $A_n \longrightarrow A$. Especially, $A_n \longrightarrow A$ does not imply $A_n \xrightarrow{c} A$.

Theorem 2. Let X be a reflexive Banach space. Let the norm $\|\cdot\|_{X^*}$ on X^* generated by the norm $\|\cdot\|_X$ on X have Property (P) (see the proof of Theorem 1). Let $(A_n) \subset \mathcal{L}(X,X)$ be such that $\|A_n\| = 1$ and $A_n \longrightarrow I$. Then $A_n^* \longrightarrow I^*$.

Proof. By the reflexivity of X, from $A_n \longrightarrow I$ it follows that $A_n^* \longrightarrow I^*$. It is

$$\|x^*\|_{X^*} \le \liminf_{n\to\infty} \|A_n^* x^*\|_{X^*} \le \limsup_{n\to\infty} \|A_n^* x^*\|_{X^*} \le \|x^*\|_{X^*}$$

for all $x^* \in X^*$. Therefore $\|A_n^* x^*\|_{X^*} \longrightarrow \|x^*\|_{X^*}$ and, by virtue of Property (P), $A_n^* \longrightarrow I^*$.

Corollary 3. Let X be a reflexive Banach space. Suppose that the norms $\|\cdot\|_X$ and $\|\cdot\|_{X^*}$ have Property

(P). Let (A_n) be a sequence of elements of $\mathcal{L}(X, X)$ such that $\| A_n \| = 1$. Then the following conditions are equivalent:

(i) $A_n \xrightarrow{c} I$. (ii) $A_n \longrightarrow I$.

(iii) $A_n \xrightarrow{c} I$. (iv) $A_n \longrightarrow I$.

(v) $A_n^* \xrightarrow{c} I^*$. (vi) $A_n^* \longrightarrow I^*$.

(vii) $A_n^* \xrightarrow{c} I^*$. (viii) $A_n^* \longrightarrow I^*$.

Remark 3. Example 1 shows that there exists a sequence (A_n) such that $A_n \longrightarrow A$ and $A_n \not\xrightarrow{c} A$. Since in the space ℓ^1 the notion of the strong convergence and the weak convergence of sequences are the same, we see (from Proposition 1) that for any Banach space Y and any $(A_n) \subset \subset \mathcal{L}(\ell^1, Y)$, $A_n \longrightarrow A$, it is $A_n \xrightarrow{c} A$. The next Theorem 3 says that in the case $\mathcal{L}(X, Y)$ where X is a separable and reflexive Banach space, this is not possible.

Theorem 3. Let X be a separable and reflexive Banach space and let Y be a Banach space. Suppose that for any sequence $(A_n) \subset \mathcal{L}(X, Y)$ such that $A_n \longrightarrow A$ it is $A_n \xrightarrow{c} A$. Then X is a finite dimensional space.

Proof. Suppose that X is an infinite dimensional space. Let $(x_n) \subset X$ be such a sequence that $Lin(x_1, \dots)$ is dense in X . We denote $X_n = Lin(x_1, \dots, x_n)$. Without loss of generality we can suppose that $X_n \subsetneqq X_{n+1}$ for any positive integer n . It is easy to see that $\overline{\bigcup_{n=1}^{+\infty} X_n} = X$.

We define by induction a sequence (e_n) such that $\|e_n\|_X = 1$

and $e_1, \dots, e_m$ is a basis for X_m and $\| e_m - y \|_X \ge \frac{1}{2}$ for each $y \in X_{m-1}$ and any integer m. (The last inequality can be guaranted for instance by using the F. Riesz theorem - see Yosida [4], Chap. III, § 2.)

According to one corollary of the Hahn-Banach theorem there exists a sequence $(f_m) \subset X^*$ such that $\langle e_i, f_m \rangle_X = \delta_{im}$, $i = 1, \dots, m$ and $\| f_m \|_{X^*} \le 2$. It is easy to see that $\langle x, f_m \rangle_X \longrightarrow 0$ for any $x \in X$. We put $y_0 \in Y$, $y_0 \neq \theta$ and we define

$$A_m : x \longrightarrow \langle x, f_m \rangle_X \, y_0$$

for any $x \in X$ and any positive integer m. Then $A_m \longrightarrow \mathcal{O}$ ($\mathcal{O}$ denotes the null operator). By the assumptions of the reflexivity, there exists a subsequence (e_{m_k}) such that $e_{m_k} \xrightarrow{X} x_0$. But $A_{m_k} e_{m_k} = \langle e_{m_k}, f_{m_k} \rangle_X y_0 = y_0 \not\longrightarrow \theta$. It shows that $A_m \overset{c}{\not\longrightarrow} \mathcal{O}$, which contradicts the assumption.

Remark 4. From the discussion of this proof we can conclude that the statement of Theorem 3 is true if X is a normed linear space with a separable and reflexive subspace of the infinite dimension and on which there exists a bounded projection P. For, if E is such a subspace, we define (A_m) on E as above. We put $B_m x = A_m P x$ for $x \in X$. Then $B_m \longrightarrow \mathcal{O}$ and $B_m \overset{c}{\not\longrightarrow} \mathcal{O}$. Unfortunately, we do not know what normed linear spaces have this property.

Corollary 4. Let X and Y be two separable and reflexive Banach spaces. Suppose that for any sequence $(A_m) \subset \mathcal{L}(X, Y)$ such that $A_m \longrightarrow A$ it is $A_m \xrightarrow{c} A$. Then X and Y are finite dimensional spaces.

Proof. By virtue of Theorem 3, X is a finite dimensional space. Using Corollary 1, Corollary 2 and Theorem 3, we obtain that Y is a finite dimensional space.

3. The convergence of projections

Theorem 4. Let X be a reflexive space and let (P_n) be a sequence of commuting projections on X, i.e. $(P_n) \subset \mathcal{L}(X,X)$, $P_n^2 = P_n$, $P_{n+1} P_n = P_n P_{n+1}$ and let $P_{n+1} P_n = P_n$, i.e. $P_n(X) \subset P_{n+1}(X)$. Then the following conditions are equivalent:

(i) $P_n \longrightarrow I$.

(ii) $P_n^* \longrightarrow I^*$.

Proof. We denote $P_n(X)$ by X_n and $P_n^*(X^*)$ by Y_n^*. By the commutativity of (P_n) and $X_n \subset X_{n+1}$, we have $Y_n^* \subset Y_{n+1}^*$. Further $Y^* = \overline{\bigcup_{n=1}^{+\infty} Y_n^*}$ is a closed convex subset of X^*. Now, we can use the Mazur theorem (see e.g. Day [2]) to get that Y^* is also weakly closed. If (i) holds then $P_n^* \longrightarrow I^*$ which shows that Y^* is a weakly dense subset of X^*. Therefore $Y^* = X^*$. From the assumption (i) it follows that $(\| P_n \|)$ is a bounded sequence and thus $(\| P_n^* \|)$ is also bounded. By the Banach-Steinhaus theorem, it remains to prove that $P_n^* x^* \longrightarrow x^*$ for any $x^* \in \bigcup_{n=1}^{\infty} Y_n^*$. But if $x^* \in Y_{n_0}^*$ then $P_n^* x^* = x^*$ for all $n \geq n_0$.

Corollary 5. Under the assumptions of Theorem 4 the following conditions are equivalent:

(i) $P_n \longrightarrow I$.

(ii) $P_n^* \longrightarrow I^*$.

(iii) $P_n \xrightarrow{c} I$.

(iv) $P_n^* \xrightarrow{c} I^*$.

<u>Remark 5.</u> The case of noncommuting projections will be obtained from Corollary 3.

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