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ON THE COMMON FIXED POINT FOR COMMUTING LIPSCHITZ FUNCTIONS

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Introduction. This note deals with the existence of solution of the equation $f(x) = g(x) = x$, where f, g are commuting and lipschitz functions.

Let f be a real-valued function defined on the set $M \subset E_1$ and $\alpha \geq 0$. f is said to be a lipschitz function on M with the constant α, if the inequality

$$|f(x) - f(y)| \leq \alpha |x - y|$$

holds for each $x, y \in M$.

Let f, g be two real-valued functions defined on the interval $I \subset E_1$ with values in I. f and g are said to be the commuting functions (we abbreviate $f \circ g = g \circ f$) if

$$f(g(x)) = g(f(x))$$

holds for each $x \in I$.

In [1] there was proved

Theorem A. Let f and g be two commuting lipschitz functions with the constants α and β, respectively, defined on $\langle 0,1 \rangle$ with values in $\langle 0,1 \rangle$.

Suppose that one of the following conditions holds:

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a) $\alpha > 1$, $\beta < \dfrac{\alpha + 1}{\alpha - 1}$,

b) $\alpha \leq 1$, $\beta \geq 0$.

Then there exists $x_0 \in \langle 0, 1 \rangle$ such that

$$f(x_0) = g(x_0) = x_0 .$$

In this note, the previous theorem will be proved for

$$\alpha > 1, \quad \beta = \frac{\alpha + 1}{\alpha - 1} .$$

Preliminary lemmas.

Lemma 1. Let f be the real-valued lipschitz mapping on an interval $I \subset E_1$ with a constant $\alpha \geq 0$. If there exist two points $x_0, y_0 \in I$, $x_0 < y_0$ such that $|f(x_0) - f(y_0)| = \alpha |x_0 - y_0|$ then f is a linear function on $\langle x_0 \, y_0 \rangle$.

Lemma 2. Let f, g be the real-valued functions defined on the interval I with values in I . Let $a \in (-\infty, \infty)$, $b \in (0, \infty)$. Denote $x^* = \dfrac{x - a}{b} = Tx$ for $x \in I$ and set

$$f^* = T \circ f \circ T^{-1} , \; g^* = T \circ g \circ T^{-1} \text{ on } I^* = T(I) .$$

The following assertions hold:

(I) $f(x) > x$ iff $f^*(x^*) > x^*$,

(II) $f(x) > g(x)$ iff $f^*(x^*) > g^*(x^*)$,

(III) $f \circ g = g \circ f$ on I iff $f^* \circ g^* = g^* \circ f^*$ on I^*,

(IV) $\dfrac{f(x)-f(y)}{x-y} = \dfrac{f^*(x^*)-f^*(y^*)}{x^*-y^*}$ for $x \neq y$.

(Proofs are obvious.)

<u>Main theorem</u>.

<u>Theorem</u>. Let f, g be two commuting mappings of any compact interval I into itself. Suppose that f and g are lipschitz functions with the constants α and β, respectively, on I.

Let $\alpha > 1$, $\beta = \dfrac{\alpha+1}{\alpha-1}$.

Then there exists $x_0 \in I$ such that

$$x_0 = f(x_0) = g(x_0) .$$

<u>Proof</u>. I. (This part of the proof and the next one are the same as a part of the proof from [1].) Suppose $\beta \geq \alpha$ and let f, g have not a common fixed point in I. Let $N_g = \{x \in I; g(x) = x\}$ and $N_f = \{x \in I; f(x) = x\}$.

It is obvious that $N_g \neq \emptyset$, $N_f \neq \emptyset$. Using the commutativity property of functions, we have $f(N_g) \subset N_g$ and $g(N_f) \subset N_f$.

Denote $a = \inf N_g$, $b = \sup N_g$. Then $a < b$ and since N_g is closed, $a, b \in N_g$. This fact implies $f(a) > a$ and $f(b) < b$.

Denote

$$x_0 = \sup\{x \in N_g;\ f(x) > x\} ,$$

$$x_1 = \inf\{x \in N_g;\ x > x_0,\ f(x) < x\} .$$

Then $x_0, x_1 \in N_g$ and

(1) $$(x_0, x_1) \cap N_g = \emptyset .$$

Evidently

(2) $$f(x_0) > x_0 , \quad f(x_1) < x_1 .$$

According to (1) we can suppose that

$$g(x) > x \quad \text{for} \quad x \in (x_0, x_1) .$$

Since $f(x_1) \in N_g - (x_0, x_1)$, $f(x_0) \in N_g - (x_0, x_1)$, we have

(3) $$f(x_1) \leq x_0 , \quad f(x_0) \geq x_1 .$$

(1,),(2) and (3) imply that the set $M = \{x \in (x_0, x_1) , \ f(x) = x\}$ is not empty and denote $s = \sup M$. Then $x_0 < s < x_1$ and $f(s) = s$.

Let $g(s) = t$. Then $t \in N_f$, $t > s$ and $t > x_1$.

II. The next relations are valid:

$$t - x_0 = g(s) - g(x_0) \leq \beta (s - x_0) ,$$

$$s - x_0 \leq s - f(x_1) = f(s) - f(x_1) \leq \alpha (x_1 - s) .$$

$$s - x_0 = \frac{\alpha}{\alpha + 1} (x_1 - x_0) ,$$

$$t - x_0 \leq f(t) - f(x_1) \leq \alpha (t - x_1) ,$$

$$\frac{\alpha}{\alpha - 1} (x_1 - x_0) \leq t - x_0 ,$$

$$t - x_0 \leq \frac{\alpha \beta}{\alpha + 1} (x_1 - x_0) \leq t - x_0 .$$

The last inequality implies

$$t - x_0 = g(s) - g(x_0) = \beta (s - x_0) ,$$

$$s - x_0 = f(s) - f(x_1) = \alpha (x_1 - s) ,$$

$$t - x_0 = f(t) - f(x_1) = \alpha(t - x_1) ,$$

$$s = \frac{\alpha}{\alpha+1}(x_1 - x_0) + x_0 , \quad t = \frac{\alpha}{\alpha-1}(x_1 - x_0) + x_0 ,$$

$$t - x_1 = g(s) - g(x_1) = \beta(x_1 - s) .$$

Hence, using Lemma 1, we have:

(4) $g(x) = \beta(x - x_0) + x_0$ for $x \in \langle x_0 , s \rangle$,

(5) $g(x) = \beta(x_1 - x) + x_1$ for $x \in \langle s , x_1 \rangle$,

(6) $f(x) = \alpha(s - x) + s$ for $x \in \langle s , x_1 \rangle$,

(7) $f(x) = \alpha(x - x_1) + x_0$ for $x \in \langle x_1 , t \rangle$.

III. We can suppose (without loss of generality) - see Lemma 2) that $s = -\alpha$ and $x_1 = \beta$.
Then $x_0 = -\alpha^2\beta$, $t = \beta^2\alpha$ and

(8) $g(x) = \beta(x + \alpha^2\beta) - \alpha^2\beta$ for $x \in \langle -\alpha^2\beta, -\alpha \rangle$,

(9) $g(x) = \beta(\beta - x) + \beta$ for $x \in \langle -\alpha , \beta \rangle$,

(10) $f(x) = \alpha(-\alpha - x) - \alpha$ for $x \in \langle -\alpha , \beta \rangle$,

(11) $f(x) = \alpha(x - \beta) - \alpha^2\beta$ for $x \in \langle \beta , \beta^2\alpha \rangle$.

Using (8), we have $g(-2\alpha - \beta) = \beta$.

The next relations are valid:

$$f(g(-2\alpha-\beta)) = -\alpha^2\beta \quad ,$$

(12) $$g(f(-2\alpha-\beta))-g(-\alpha) = f(g(-2\alpha-\beta))-g(-\alpha) = -\alpha^2\beta - \beta^2\alpha \quad ,$$

$$|g(f(-2\alpha-\beta)) - g(-\alpha)| \leq \beta|f(-2\alpha-\beta)+\alpha| \ .$$

Using $f(-\alpha) = -\alpha$, we obtain

(13) $$|f(-2\alpha-\beta) - f(-\alpha)| \geq \alpha\beta + \alpha^2 \ .$$

But

(14) $$|f(-2\alpha-\beta) - f(-\alpha)| \leq \alpha|\alpha+\beta| \ .$$

From (13) and (14) we obtain:

f is a linear function on $\langle -2\alpha-\beta, -\alpha \rangle$ and $|f(x) - f(-\alpha)| = \alpha|x+\alpha|$.

After a simple calculation we obtain that $f(-2\alpha-\beta) = -\alpha^2\beta$ is not possible. Thus

(15) $$\begin{cases} f(x) = \alpha(-\alpha-x) - \alpha \text{ for } x \in \langle -2\alpha-\beta, -\alpha \rangle & \text{and} \\ f(-2\alpha-\beta) = \alpha^2 + \alpha\beta - \alpha \ . \end{cases}$$

According to (12) we have

(16) $$\alpha^2\beta + \beta^2\alpha = -(g(\alpha^2+\alpha\beta-\alpha) - g(-\alpha)) \ ,$$

and

(17) $$|g(\alpha^2+\alpha\beta-\alpha) - g(-\alpha)| \leq \beta|\alpha^2+\alpha\beta| \ .$$

Hence, using Lemma 1, it is

$$(18)\quad g(x) = \beta(-\alpha - x) + \beta^2\alpha \ \text{for} \ x \in \langle -\alpha, \alpha^2 + \alpha\beta - \alpha \rangle .$$

Similarly as in (15),(18), we obtain

$$(19)\quad g(x) = -\beta(x - \beta) + \beta \ \text{for} \ x \in \langle \beta, 2\beta + \alpha \rangle ,$$

$$(20)\quad f(x) = \alpha(\beta - x) - \alpha^2\beta \ \text{for} \ x \in \langle -\beta^2 - \alpha\beta + \beta, \beta \rangle .$$

IV. In the previous parts of this proof we proved under assumption f and g have not a common fixed point that the relations (8) - (20) are valid. In the next step we show that it is not possible.

Suppose, for example $\beta > 3$.

Then $\beta - \alpha\beta - \beta^2 < -\alpha^2\beta$ and

(20)implies $f(-\alpha^2\beta) = \alpha^3\beta + \alpha\beta - \alpha^2\beta$,

(19) implies $g(2\beta + \alpha) < -\alpha^2\beta$,

$$g(\alpha^3\beta + \alpha\beta - \alpha^2\beta) = g(f(-\alpha^2\beta)) =$$
$$= f(g(-\alpha^2\beta)) = \alpha^3\beta + \alpha\beta - \alpha^2\beta$$

and thus

$$\alpha^3\beta + \alpha\beta < (g(\alpha^3\beta + \alpha\beta - \alpha^2\beta) - g(2\beta + \alpha)) =$$
$$= \beta|\alpha^3\beta + \alpha\beta - \alpha^2\beta - 2\beta - \alpha| ,$$
$$\alpha(\alpha - 1) < |\alpha^2 - 2| .$$

The last inequality is not true for $\beta > 3$.

Suppose $2 \leq \alpha \leq 3$, $2 \leq \beta \leq 3$. The relations (11) and (18) imply

$$f(\alpha^2 + \alpha\beta - \alpha) = \alpha^3 - \alpha^2 - \alpha\beta ,$$

$$g(\alpha^3-\alpha^2-\alpha\beta) = -\beta\alpha + 2\beta^2\alpha + \beta\alpha^2 - \beta\alpha^3 .$$

Thus

$$f(-\alpha^2\beta) = f(g(\alpha^2+\alpha\beta-\alpha)) = g(f(\alpha^2+\alpha\beta-\alpha) =$$

$$= -\beta\alpha + 2\beta^2\alpha + \beta\alpha^2 - \beta\alpha^3 ,$$

$$(21)\quad |f(-\alpha^2\beta)-f(\beta-\alpha\beta-\beta^2)| = \beta\alpha(\alpha^2+1-\alpha-\beta) .$$

$$(22)\quad |f(-\alpha^2\beta)-f(\beta-\alpha\beta-\beta^2)| \leq \alpha|\alpha^2\beta-\alpha\beta+\beta-\beta^2| .$$

From (11),(22) and Lemma 1 we have

$$(23)\quad \begin{aligned} &f(x) = -\alpha(x+\alpha\beta+\beta^2-\beta)+\beta^2\alpha \\ &\text{for } x \in \langle -\alpha^2\beta, \beta-\alpha\beta-\beta^2 \rangle \end{aligned}$$

and similarly

$$(24)\quad \begin{aligned} &g(x) = \beta(x-\alpha^2-\alpha\beta+\alpha)-\alpha^2\beta \\ &\text{for } x \in \langle \alpha^2+\alpha\beta-\alpha, \beta^2\alpha \rangle . \end{aligned}$$

It is easy to show that under assumption that the relations (8),(9),(10),(11),(15),(18),(19),(20),(23),(24) are valid, f, g are not commuting.

The proof is completed.

Remarks: P. Huneke in [2] proved that in the case $\alpha = \beta > 3 + \sqrt{6}$ the problem about common fixed point for the commuting and lipschitz functions has no solution in general.

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R e f e r e n c e s

[1] R. de MARR: A common fixed point for commuting mappings, Amer.Math.Monthly 70(1963),535-537.

[2] P. HUNEKE: On common fixed points of commuting functions on an interval, Trans.Amer.Math.Soc. 139(1969),371-381.

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