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A SEPARATION THEOREM FOR FINITE FAMILIES

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It is widely recognized that separation of two convex sets by linear functionals plays an important role in the theory of optimization. Recently, beginning with the paper by Dubovitskii and Milyutin [1], it has been convincingly demonstrated by a number of authors (in addition to [1] see, as examples, [2], [3], [4] and [5]) that the separation properties of families of convex sets also represent natural and important tools in the theory of optimization in linear spaces. In this note a basic separation theorem for finite families of convex sets in real linear spaces is presented.

Let L be a real linear space and let I be a finite set. We say that a family $\{Q_i : i \in I\}$ of subsets of L can be separated if there is a family $\{f_i : i \in I\}$ of linear (i.e. additive and homogeneous) functionals on L and a family $\{\lambda_i : i \in I\}$ of real numbers such that

(1) f_i is not identically zero for some $i \in I$,

(2) $Q_i \subset \{x \in L \mid f_i(x) \leqq \lambda_i\}$ whenever $i \in I$,

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(3a) $\sum_{i \in J} f_i = 0$, (3b) $\sum_{i \in J} \lambda_i \leqq 0$.

Note that the intersection $\bigcap_{i \in J} \{x \in L \mid f_i(x) < \lambda_i\}$ is empty provided (3a) and (3b) are satisfied.

For subsets A and B of L (following the terminology and notation of Victor Klee [6]), the core of A relative to B , denoted $cr_B(A)$, is defined as follows: $a \in cr_B(A)$ if and only if for each element b of B there is a positive real number α such that A contains the segment $[a, a + \alpha(b - a)]$. The core of A relative to the affine hull of A is called the intrinsic core of A and is denoted by $icr(A)$.

Lemma. If M is a convex subset of a real linear space L such that $icr(M)$ is nonempty and such that the zero-element of L does not belong to $icr(M)$, then there is a linear functional on L such that $f(x) \leqq 0$ whenever $x \in M$.

Proof. Let us consider the set $K_M = \{x \in L \mid x = \alpha y, \alpha \geqq 0, y \in M\}$. The set K_M is a pointed convex cone (with vertex 0) different from L and the core of K_M relative to the linear hull of K_M is nonempty (since the intrinsic core of M is nonempty). It implies (see [7], Chapter 1,Theorem 3.2) that there is a not identically zero linear functional f' on the linear hull of K_M such that $f'(x) \leqq 0$ whenever $x \in K_M$ and thus also $f'(x) \leqq 0$ whenever $x \in M$. Extending f' linearly to the whole of L we obtain a linear functional f on L with the required property.

Theorem. If I is a finite set and if a family $\{Q_i : i \in I\}$ of subsets of a real linear space L satisfies the conditions

(a) Q_i is convex for each $i \in I$,

(b) $icr(Q_i)$ is nonempty for each $i \in I$,

(c) $\bigcap_{i \in I} icr(Q_i)$ is empty,

then the family $\{Q_i : i \in I\}$ can be separated.

Proof. It is a simple exercise in elementary logic to verify that the theorem is valid for the empty family and for the families consisting of one set only. Further let us notice (considering $I = \{0, 1\}$, $Q_0 = \{0\}$, $Q_1 = M$, $f_0 = -f$, $f_1 = f$, $\lambda_0 = \lambda_1 = 0$) that the lemma expresses the fact that the family $\{\{0\}, M\}$ can be separated. Since for two subsets A_0 and A_1 of L

(α) A_0, A_1 can be separated if and only if $\{\{0\}, A_0 - A_1\}$ can be separated,

(β) $A_0 - A_1$ is convex if both A_0 and A_1 are convex,

(γ) $icr(A_0 - A_1)$ is nonempty if both $icr(A_0)$ and $icr(A_1)$ are nonempty.

(δ) $0 \notin icr(A_0 - A_1)$ if $icr(A_0)$ and $icr(A_1)$ are disjoint,

the lemma ensures that the theorem is valid for families consisting of two sets. It remains to consider the families consisting of more than two sets. Since any finite set I with more than two elements will serve as well as any other for the purposes of this proof, suppose for the sake of definiteness that $I = \{0, 1, \dots, n\}$, $n > 1$.

Let us consider the Cartesian product $P = \prod_{i=1}^{m} L_i$, where $L_i = L$ for $i = 1, 2, \ldots, m$. The sets

$M = \{ y \in P \mid y = (x_1, x_2, \ldots, x_m)$, $x_i \in Q_i$ for $i = 1, 2, \ldots, m \}$,

$N = \{ y \in P \mid y = (x_1, x_2, \ldots, x_m)$, $x_1 = x_2 = \ldots = x_m = x$ for some $x \in Q_0 \}$

are convex and in addition both $icr(M)$ and $icr(N)$ are nonempty and the intersection $icr(M) \cap icr(N)$ is empty. Hence there are linear functionals g, h on P and real numbers λ, μ satisfying

$$g \neq 0 \quad \text{or} \quad h \neq 0 ,$$

$g(y) \leq \lambda$ whenever $y \in M$ and $h(y) \leq \mu$ whenever $y \in N$,

$$g + h = 0, \quad \lambda + \mu \leq 0 .$$

Since the space P^* is isomorphic to the space $\prod_{i=1}^{m} L_i^*$, where P^* (and similarly L_i^*) denotes the space of all linear functionals on P , there are linear functionals $f_1, f_2, \ldots, f_m$ on L such that for each $y = (x_1, x_2, \ldots, x_m)$ of P

$$g(y) = \sum_{i=1}^{m} f_i(x_i) .$$

Defining $f_0 = - \sum_{i=1}^{m} f_i$, we obtain a family $\{ f_i : i \in I \}$ of linear functionals on L such that (1) and (3a) are satisfied. Since

$$g(y) = \sum_{i=1}^{m} f_i(x_i) \leq \lambda$$

whenever $(x_1, x_2, \dots, x_m) \in M$, there are real numbers $\lambda_1, \lambda_2, \dots, \lambda_m$ such that $f_i(x_i) \leqq \lambda_i$ whenever $i \in \{1, 2, \dots, m\}$ and $x_i \in Q_i$ and such that $\sum_{i=1}^{m} \lambda_i \leqq \lambda$. Denoting μ by λ_0 we obtain a family $\{\lambda_i : i \in I\}$ of real numbers such that in addition to (1) and (3a) also (2) and (3b) are satisfied. This completes the proof.

If in addition to the assumptions of the theorem all the sets Q_i are cones with the vertex 0 , then all λ_i appearing in (2) must be zero. Indeed, if $\lambda_i \neq 0$ for some $i \in I$, then by (3b) $\lambda_j < 0$ for some $j \in I$ and taking $x = \alpha y$, where $y \in Q_j$ and $\alpha = \frac{\lambda_j}{2f_j(y)}$,we obtain an element of Q_j contradicting the property (2). This proves the following

<u>Corollary</u>. If J is a finite set and if a family $\{Q_i : i \in I\}$ of subsets of a real linear space satisfies the conditions

(a′) Q_i^* is a convex cone with the vertex 0 for each $i \in I$,

(b) $icr(Q_i)$ is nonempty whenever $i \in I$,

(c) $\bigcap_{i \in J} icr(Q_i)$ is empty, then there is a family $\{f_i : i \in I\}$ of linear functionals on L such that

(1) f_i is not identically zero for some $i \in I$,

(2′) $f_i(x) \leqq 0$, whenever $i \in I$ and $x \in Q_i$,

(3a) $\sum_{i \in J} f_i = 0$.

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