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VECTOR BUNDLES AS AN INSTRUMENT OF THE METRIC AND CONFORMAL DIFFERENTIAL GEOMETRY

(Preliminary communication)

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In the following we shall give an abstract of the author's papers [5] and [6] (see references at the end of this note).

I. Submanifolds in a space of constant curvature

In [3] and [4] we have constructed a vector bundle model of a manifold M immersed into a space N of constant curvature. In the present paper [5] we use this model for a global formulation and generalization of some results by C.B. Allendoerfer concerning type numbers (cf. [1]).

Let us remind the basic definitions.

<u>A graded Riemannian vector bundle</u> $\{E^k, P_k\}^r$ <u>over a manifold</u> M is a Riemannian vector bundle $E \longrightarrow M$, $\dim E \geq \dim M$, in which the following structure is given:
(i) a fixed bundle injection $j : T(M) \longrightarrow E$,
(ii) an orthogonal splitting (graduation) $E = E^1 \oplus \ldots \oplus E^r$ such that $E^1 \equiv j\,T(M)$ (E^1 will be identified with

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$T(M)$),

(iii) a system of bundle epimorphisms

$$P_k : E^1 \otimes E^k \longrightarrow E^{k+1}, \quad k = 1,\dots,n-1 ,$$

such that the composed mappings

$$P^k(X_1,\dots,X_k) = (P_{k-1} \circ \dots \circ P_2 \circ P_1)(X_1 \otimes \dots \otimes X_k)$$

are all symmetric.

We define <u>dual homomorphisms</u> $L_k : E^1 \otimes E^k \longrightarrow E^{k-1}$, $k = 2,\dots,n$, by means of the formula

$$(1) \quad \langle L_k(T \otimes X^{(k)}), Y^{(k-1)} \rangle = - \langle X^{(k)}, P_{k-1}(T \otimes Y^{(k-1)}) \rangle .$$

Here $X^{(k)}$ denotes a section of M into E^k. We write simply $P_k(T, X^{(k)})$, $L_k(T, X^{(k)})$ instead of $P_k(T \otimes X^{(k)})$, $L_k(T \otimes X^{(k)})$ in the following.

By a <u>sequence of canonical connections in</u> $\{E^k, P_k\}^n$ we mean a sequence of linear connections $\nabla^{(1)},\dots,\nabla^{(n)}$ in the vector bundles $E^1,\dots,E^n$ respectively such that

(i) each $\nabla^{(k)}$ preserves the inner product in E^k,

(ii) $\nabla^{(1)}$ is the canonical Levi-Civita connection in $E^1 \equiv T(M)$,

(iii) <u>the Codazzi equation</u>

$$(2) \quad \begin{aligned} &\nabla_u^{(k+1)} P_k(T, X^{(k)}) - \nabla_T^{(k+1)} P_k(u, X^{(k)}) + P_k(u, \nabla_T^{(k)} X^{(k)}) - \\ &- P_k(T, \nabla_u^{(k)} X^{(k)}) - P_k([u, T], X^{(k)}) = 0 \end{aligned}$$

holds for $k = 1,\dots,n-1$.

Remark that if such a sequence exists in $\{E^k, P_k\}^n$, then it is unique.

Let us denote by $R^{(k)}$ the curvature transformation of the connection $\nabla^{(k)}$. <u>The Gaussian equation with the</u>

parameter C and of order k is given by

$$\begin{aligned} &R^{(k)}_{uT} X^{(k)} + P_{k-1}(u, L_k(T, X^{(k)})) - P_{k-1}(T, L_k(u, X^{(k)})) + \\ (3) \qquad &+ L_{k+1}(u, P_k(T, X^{(k)})) - L_{k+1}(T, P_k(u, X^{(k+1)})) = \\ &= C\{\langle T, X^{(k)}\rangle u - \langle u, X^{(k)}\rangle T\} \quad (k = 1, \dots, \varkappa). \end{aligned}$$

A Riemann geometry $G_{\varkappa,C}$ of genus $\varkappa$ and with the exterior curvature C on a manifold M is a graded Riemannian vector bundle $E = \{E^k, P_k\}^{\varkappa}$ over M such that

(i) a sequence $\nabla^{(1)}, \dots, \nabla^{(\varkappa)}$ of canonical connections exists in E,

(ii) the Gaussian equations (3) hold for $k = 1, \dots, \varkappa - 1$.

A Riemannian geometry $G_{\varkappa,C}$ is called integrable if the $\varkappa$-th Gaussian equation holds, too.

The relationship between Riemannian geometries (particularly maximal Riemannian geometries) and immersions of manifolds into space forms is studied in [3],[4].

Now, a Riemannian geometry $G_{\varkappa,C} = \{E^k, P_k\}^{\varkappa}$ is called of type $t \geq k$ ($k = 0, 1, \dots$) if the bundle morphism $L_\varkappa : E^1 \otimes E^{\varkappa} \longrightarrow E^{\varkappa-1}$ has the following property at each point $x \in M$: there is a k-dimensional subspace $F_x \subset E^1_x$ such that the restricted map $L_{\varkappa,x} : F_x \otimes E^{\varkappa}_x \longrightarrow E^{\varkappa-1}_x$ is injective. The following global theorems are proved in [5]:

T1. Any Riemannian geometry $G_{\varkappa,C}$ of type $t \geq 3$ is integrable.

T2. Any two prolongations $G_{\varkappa+1,C}$, $G'_{\varkappa+1,C}$ of type $t \geq 3$ of the same Riemannian geometry $G_{\varkappa,C}$ are equivalent.

T3. If $E = \{E^k, P_k\}^n$ is a graded Riemannian vector bundle of type $t \geq 4$ such that a sequence $\nabla^{(1)}, \ldots, \nabla^{(k-1)}$ of canonical connections exists in the graded subbundle $\{E^k, P_k\}^{k-1}$, then the last canonical connection $\nabla^{(k)}$ exists provided that the Gaussian equation of order $k-1$ holds.

II. Submanifolds of a conformally euclidean space

A. Fialkow [2], has characterized a submanifold of a conformally euclidean space N by a number of tensors, called conformal fundamental tensors, exact up to a conformal transformation of N. In [6] we develop a more elegant theory, which enables to characterize a submanifold $M \subset N$ by a canonical structure of the induced bundle $\varphi_* T(N)$ ($\varphi : M \longrightarrow N$ is the inclusion map).

Basic definitions. A Riemannian bundle $E(A,\nabla) \longrightarrow M$ is a vector bundle $E \longrightarrow M$ provided with a fibre metric A and with a linear connection ∇ preserving the inner product A.

A bundle $E(A,\nabla) \longrightarrow M$, $\dim E \geq \dim M$, is called soldered if there is given a fixed bundle injection $j : T(M) \longrightarrow E$ such that ∇ is torsion-free with respect to j, i.e., such that $\nabla_U j(T) - \nabla_T j(U) - j([U,T]) = 0$ for any vector fields U, T on M. We consider the tangent bundle $T(M)$ as a Riemannian subbundle $T(M)(A, \nabla^\tau)$ of $E(A,\nabla)$, where ∇^τ is the orthogonal projection of the connection ∇ into $T(M)$. Here A defines a Riemann metric on M and ∇^{τ} is the corresponding Levi-Civita connection.

Now, for any soldered Riemannian vector bundle $E(A,\nabla) \longrightarrow M$, $\dim M \geq 3$, we can define a bundle morphism $C: T(M) \otimes T(M) \longrightarrow Hom(E,E)$, called <u>the Weyl transformation</u>, and a bundle morphism $D: T(M) \longrightarrow E$, called the <u>deviation transformation</u>.

<u>Basic result</u>: (Generalized Schouten's theorem)

<u>Let</u> $E(A,\nabla) \longrightarrow M$ <u>be a soldered Riemannian vector bundle</u>, $\dim M \geq 3$.

<u>If and only if</u>

a) $C = 0$ <u>in the case</u> $\dim M \geq 4$, <u>or</u>

b) $C = 0$, $(\nabla_u D)(V) - (\nabla_V D)(U) = 0$ <u>in the case</u> $\dim M = 3$, <u>the bundle</u> $E(A,\nabla)$ <u>is locally conformally euclidean in the following sense:</u> there is a conformal imbedding φ of a neighbourhood U of <u>any</u> point $p \in M$ into a conformally euclidean space N such that the induced bundle $\varphi_* T(N)$ is "conformally equivalent" to $E(A,\nabla)_{|U}$. The imbedding φ can be determined uniquely by the addition of a system of initial conditions. Any two imbeddings φ, φ' of U into N correspending to different systems of initial conditions can be transformed one into another by a local conformal transformation F of the space N.

In case that $E \equiv T(M)$ we obtain hence the classical Schouten's theorem.

R e f e r e n c e s

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