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ON HOMOMORPHISM PERFECT GRAPHS

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Introduction. The homomorphism graph is defined as a graph which arises in a natural way on the set of all endomorphisms of a graph. Here we are interested in the question, under which conditions a graph is isomorphic with its homomorphism graph.

We shall need the following definitions:

Let X be a set. Let M be a set of mappings of X into itself. The pair (X, M) is called a transformation monoid if the identity mapping of X belongs to M and the set M is closed under composition.

Two transformation monoids (X, M) and (Y, N) are isomorphic if there exists a 1-1 mapping $F: X \rightarrow Y$ such that the mapping $\mathcal{F}: M \rightarrow N$ defined by $\mathcal{F}(f)(F(x)) = F(f(x))$ is an algebraic isomorphism of monoids (M, N) .

A transformation monoid (X, M) is called abstract if (X, M) is isomorphic with (M, L_M) where $L_M = \{L_f \mid f \in M\}$ ($L_f: M \rightarrow M$ is defined by: $L_f(g) = f \circ g$).

We shall use the following well-known lemma (see

e.g. [4]).

Lemma. A transformation monoid (X, M) is abstract if and only if there exists $x_0 \in X$ such that for every $x \in X$ there exists exactly one $f \in M$ such that $f(x_0) = x$ (x_0 is called an exact source).

A graph (X, R) is a set X with relation $R \subset X \times X$. Concerning graphs we use the notations of [1]. Let us remark that the monoid $C(X, R)$ of all compatible mappings (homomorphisms) of the graph (X, R) into itself is understood here in its actual form as a transformation monoid.

All the graphs concerned here are finite.

The following definition was suggested by Z. Hedrlín.

Definition. Let (X, R) be a graph. Define the homomorphism graph $(C(X, R), M)$ of the graph (X, R) as follows:

$f, g \in C(X, R)$, then $(f, g) \in M \iff (f(x), g(x)) \in R$ for every $x \in X$. Note that this graph is one of the graphs related to tensor products, see [2]. We say that the graph (X, R) is homomorphism perfect if it is isomorphic to the graph $(C(X, R), M)$. The property of being homomorphism perfect is studied here in its relationship to the abstractness of the transformation monoid $(X, C(X, R))$.

Theorem 1. Let (X, R) be a homomorphism perfect graph. Then the transformation monoid $(X, C(X, R))$ is abstract.

Proof. Let F be an isomorphism of (X, R) onto $(C(X, R), M)$ - the homomorphism graph of (X, R) .

Consider $C(C(X, R), M)$. Clearly,
 $L_f \in C(C(X, R), M)$ for every $f \in C(X, R)$ ($(f_i, f_j) \in M$
 $\in M$ implies ${}^2L_f(f_i, f_j) = (f \circ f_i, f \circ f_j) \in M$ as
 f is compatible). Further, $f_i \neq f_j$ implies $L_{f_i} \neq$
 L_{f_j} .

Thus $\text{card } C(X, R) = \text{card } C(C(X, R), M) = \text{card } L_M$.
 Since $L_M \subset C(C(X, R), M)$, $L_M = C(C(X, R), M)$ holds.
 Thus it remains to prove that the transformation monoids
 $(X, C(X, R))$ and $(C(X, R), C(C(X, R), M))$ are iso-
 morphic.

We shall show that this isomorphism is carried by
 the mapping F , i.e. that the mapping \mathcal{F} defined by
 $\mathcal{F}(f)(F(x)) = F(f(x))$ is an algebraic isomorphism.

First, we shall prove that $f \in C(X, R)$ implies
 $\mathcal{F}(f) \in C(C(X, R), M)$. Let $(f_i, f_j) \in M$. Then
 $(\mathcal{F}(f)(f_i), \mathcal{F}(f)(f_j)) = (\mathcal{F}(f)F(F^{-1}(f_i)),$
 $\mathcal{F}(f)F(F^{-1}(f_j))) = (F(f(F^{-1}(f_i))), F(f(F^{-1}(f_j)))) \in M$.

Evidently $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$ for every f, g .
 Further, $f \neq g$ implies $\mathcal{F}(f) \neq \mathcal{F}(g)$ and consequent-
 ly \mathcal{F} is 1-1. Q.E.D.

Theorem 1 does not give a sufficient condition for
 homomorphism perfect graphs. We construct a graph (even
 a class of graphs) possessing an abstract transformation
 monoid of homomorphisms into itself which is not a homo-
 morphism perfect graph.

Example. Let n be an even number. Define the graph
 (X_n, R) by $X_n = \{1, \dots, n\}$ and $R = \{(1, 1), (2, 2),$

$(3, 3), \dots, (\frac{m}{2}, \frac{m}{2}), (\frac{m}{2} + 1, \frac{m}{2} + 2), (\frac{m}{2} + 3, \frac{m}{2} + 3), \dots$
 $\dots, (m-1, m), (m, \frac{m}{2} + 1), (1, \frac{m}{2} + 1), (2, \frac{m}{2} + 2), \dots, (\frac{m}{2}, m)\}$.

Evidently $C(X, R) = \{c_1, \dots, c_{\frac{m}{2}}, id, f, f^2, \dots, f^{m-1}\}$,
 where $c_i(j) = i$ for all $j = 1, \dots, m$; $f(i) = i + 1$ for
 $i \neq \frac{m}{2}, m$; $f(\frac{m}{2}) = 1, f(m) = \frac{m}{2} + 1$. Clearly $f^m = id$.
 Let F be an isomorphism of (X, R) onto $(C(X, R), M)$.
 We have $(c_i, c_i) \in M$ for all $i = 1, \dots, \frac{m}{2}$, thus
 $F\{1, \dots, \frac{m}{2}\} = \{c_1, \dots, c_{\frac{m}{2}}\}$. Thus there exists an i ,
 $\frac{m}{2} \leq i \leq m$ such that $F(i) = id$, therefore
 $(c_{i-\frac{m}{2}}, id) \in M$, i.e. $(i - \frac{m}{2}, j) \in R$ holds for every $j =$
 $= 1, \dots, m$. This is a contradiction. (Evidently $C(X, R)$
 is abstract monoid, any $i = \frac{m}{2} + 1, \dots, m$ can serve as
 an exact source.)

In the following theorem we give a sufficient condi-
 tion for a graph to be homomorphism perfect.

Theorem 2. Let (X, R) be a graph. If the trans-
 formation monoid $(X, C(X, R))$ is abstract and commu-
 tative, then the graph (X, R) is homomorphism perfect.

Proof. There exists an $x_0 \in X$ which is not an ex-
 act source of $(X, C(X, R))$. Define the mapping
 $F: X \rightarrow C(X, R)$ by $F(x) = f$, where $f(x_0) = x$
 (such f is determined uniquely). We shall prove that F
 is an isomorphism of (X, R) onto $(C(X, R), M)$. Evi-
 dently F is 1-1.

Let $(x_1, x_2) \in R$ and let $F(x_i) = f_i, i = 1, 2$.

Then

$$(F(x_1)(x), F(x_2)(x)) = (f_1(x), f_2(x)) = (f_1(F(x)(x_0)), f_2(F(x)(x_0))) = (F(x)(f_1(x_0)), F(x)(f_2(x_0))) = (F(x)(x_1), F(x)(x_2))$$

and as F is a compatible mapping $(F(x)(x_1), F(x)(x_2)) \in R$ for all $x \in X$. Thus $(F(x_1), F(x_2)) \in M$. Let $(f_1, f_2) \in M$. Putting $x = x_0$ we get $(x_1, x_2) \in R$,

Q.E.D.

A trivial consequence follows from the last part of our proof: Let (X, R) be a graph, $(C(X, R), M)$ its homomorphism graph. If the transformation monoid $(X, C(X, R))$ is abstract, then the graph $(C(X, R), M)$ is isomorphic with a spanning subgraph of (X, R) .

Now, the functional graphs will be studied. A graph (X, R) is called functional if for every $x \in X$ there exists at most one $y \in X$ such that $(x, y) \in R$ (see e.g. [31]).

Let k, m be integers, $1 \leq k \leq m$. Define the graph $G_{k,m} = (X_m, R)$ by $X_m = \{1, \dots, m\}$, $R = \{(1, 2), (2, 3), \dots, (m-1, m), (m, k+1)\}$. Evidently $G_{k,m}$ is functional.

Theorem 3. Let (X, R) be a functional graph. (X, R) is homomorphism perfect if and only if there are integers k and m ($1 \leq k \leq m$) such that $(X, R) = G_{k,m}$.

Proof. We shall prove that $G_{k,m}$ is homomorphism perfect. Evidently $C(G_{k,m}) = \{id, f, f^2, \dots, f^{m-1}\}$,

where f is defined by $f(i) = i + 1, i = 1, \dots, n-1; f(n) = k$. Since $(X_m, C(G_{k,m}))$ is abstract (1 is an exact source) and commutative, $G_{k,m}$ is a homomorphism perfect graph. Let (X, R) be a functional graph $(X, R) \neq G_{k,m}$ for any k, m . We shall prove that $(X, C(X, R))$ is not abstract, i.e. (X, R) is not homomorphism perfect. Suppose that (X, R) is abstract. Let x_0 be an exact source. Let x_1, \dots, x_r be all points of X such that there does not exist $(x_i', x_i) \in R, i = 1, \dots, r$. Clearly $r \geq 2$. Evidently, $x_0 \in \{x_1, \dots, x_r\}$. For every $i = 1, \dots, r$ there exist k_i, m_i such that G_{k_i, m_i} is a subgraph of (X, R) . ($G_{k_i, m_i} = (\{x_1, x_2, \dots, x_{m_i}\}, \{(x_1, x_2), \dots, (x_{m_i-1}, x_{m_i}), (x_{m_i}, x_{k_i+1})\})$ where $x_1 = x_1$.) Evidently, $\{x_{k_i+1}, x_{k_i+2}, \dots, x_{m_i}\} = \bar{X}$ ($\text{card } \bar{X} = k$) is the same set for all $i = 1, \dots, r$. (If the opposite holds then there exist two sets $A \subset X, B \subset X$ such that $A \cap B = \emptyset, A \cup B = X$ such that $(a, b) \notin R$ and $(b, a) \notin R$ for every $a \in A, b \in B$. Assume that $x_0 \in A$. Then $(B, R/B)$ must be rigid (see [1]). This is a contradiction.)

Denote the points of \bar{X} by $\bar{x}_1, \dots, \bar{x}_k$. For every x_i ($i = 1, \dots, r$) there exists t ($1 \leq t \leq k$) such that $x_{k_i+1} = \bar{x}_t$. We say that x_i belongs to \bar{x}_t . Suppose that there exist x_{r_1}, \dots, x_{r_m} ($m \geq 2, 1 \leq r_1 < r_2 < \dots < r_m \leq r$) such that x_{r_1}, \dots, x_{r_m} belongs to the same \bar{x}_t . Let $k_t = \max(k_{r_1}, \dots, k_{r_m})$.

We shall prove $x_n \neq x_0$. Let us define f_0 as follows:

$$f_0(i x_{n_i}) = i x_{n_i}, f_0(i x_{n_i-1}) = i x_{n_i-1}, \dots, f_0(i x_1) = i x_{n_i-n_i+1}$$

for all $i = r_1, \dots, r_m$ and $f_0 = id$ for the rest of X .

Evidently, f_0 is an homomorphism and since $id(x_n) =$

$$= x_n, f_0(x_n) = x_n, x_n \text{ is not an exact source.}$$

Clearly there holds: $n_j < n_k, j \in \{r_1, \dots, r_m\}$ implies x_j is not an exact source as there is no homomorphism f such that $f(x_j) = x_n$. The mapping f_0 also shows that no x_j ($j \in \{1, \dots, n\} \setminus \{r_1, \dots, r_m\}$) is an exact source.

Hence it follows that for every \bar{x}_t ($t = 1, \dots, n$) there exists at most one x_i ($i = 1, \dots, r$) belonging to \bar{x}_t . Suppose that x_j ($j \in \{1, \dots, r\}$) is an exact source (i.e. $x_j = x_0$). Let $n_i = q_i n + n_i$ where q_i, n_i are positive integers $n_i < n, n = n_i - n_i, i = 1, \dots, r$. Define f' by:

$$f'(i x_{n_i}) = i x_{n_i}, f'(i x_{n_i-1}) = i x_{n_i-1}, \dots,$$

$$f'(i x_{2n_i-m_i}) = i x_{n_i}, \dots,$$

$$\dots \dots \dots \dots \dots \dots,$$

$$f'(i x_{q_i+1 n_i - q_i m_i}) = i x_{n_i}, \dots, f'(i x_1) = i x_{n_i - n_i + 1}$$

for all $i \neq j$ and $f' = id$ for the rest of X .

Evidently, f' is a homomorphism. Since $id(x_j) = x_j$

$$f'(x_j) = x_j, \text{ we have a contradiction. This proof}$$

holds for all $j = 1, \dots, r$, hence (X, R) is not

abstract, Q.E.D.

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R e f e r e n c e s

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