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UPPER SEMICOMPLEMENTS AND A DEFINABLE ELEMENT IN THE LATTICE OF GROUPOID VARIETIES

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The variety of semigroups is not generated by any finite number of its proper subvarieties (see Dean and Evans [2]). An analogous statement holds for the lattices of varieties of groups, lattices, loops and commutative semigroups (see Evans [3] for the summary and bibliography). It is proved in [6] that this property is not shared by the variety of all universal algebras of a given type Δ containing at least one at least binary function symbol: there are found in the lattice $\mathcal{L}_\Delta$ of varieties of algebras of type Δ some upper semicomplements different from the greatest element ι_Δ of $\mathcal{L}_\Delta$. In the present paper we shall restrict ourselves to the case of the lattice $\mathcal{L}_\Gamma$ of groupoid varieties and investigate upper semicomplements in $\mathcal{L}_\Gamma$.

In § 2 the infimum of the set of all upper semicomplements in $\mathcal{L}_\Gamma$ is found: it is just the variety of commutative semigroups satisfying $x^2 . y = x . y$. This variety is thus a definable element in $\mathcal{L}_\Gamma$.

To prove the result, we must find some further upper

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semicomplements in $\mathcal{L}_\Gamma$. These are found in § 1.

For the terminology and notation see [6] and § 1 of [4].

§ 1. Some upper semicomplements in $\mathcal{L}_\Gamma$

We denote by Γ the type of groupoids, i.e. the type consisting of a single binary function symbol. The terminology given in [4] and [6] can be specialized to the case $\Delta = \Gamma$: e.g. W_Γ denotes the free groupoid freely generated by X . Γ -equations are called equations throughout the paper, etc. If u and v are two elements of W_Γ , then the value of the fundamental binary operation of W_Γ , applied to u and v , is denoted by $u \cdot v$ or only uv . We write $uv \cdot w$ instead of $(u \cdot v) \cdot w$, etc.

For every $t \in W_\Gamma$ we define two elements $\overleftarrow{t}$ and $\overrightarrow{t}$ of W_Γ in this way: if $t \in X$, then $\overleftarrow{t} = \overrightarrow{t} = t$; if $t = t_1 \cdot t_2$, then $\overleftarrow{t} = t_1$ and $\overrightarrow{t} = t_2$.

For every $t \in W_\Gamma$ we define elements $\sigma_1(t)$, $\sigma_2(t), \sigma_3(t), \ldots$ of W_Γ in this way: $\sigma_1(t) = tt \cdot t$; $\sigma_{n+1}(t) = (\sigma_n(t) \cdot \sigma_n(t)) \cdot \sigma_n(t)$.

Let us fix two different variables (i.e. elements of X) and denote them by x_0 and y_0 . Put

$$e_1 = \langle (x_0 x_0 \cdot x_0) y_0, x_0 y_0 \rangle ; \quad e_2 = \langle (x_0 \cdot x_0 x_0) x_0, x_0 x_0 \rangle ;$$

$$e^1 = \langle x_0 x_0 \cdot x_0, x_0 x_0 \rangle ; \quad e^2 = \langle x_0 \cdot x_0 x_0, x_0 x_0 \rangle .$$

Let e be any of the four equations e_1, e_2, e^1 and e^2 . It will be useful to notice that the following (trivial) assertion holds: whenever u_1, u_2, v_1 and v_2 are elements of W_Γ such that $v_1 v_2$ is a leap-consequence of $u_1 u_2$ by means of e, then no one of the three cases

(i) $u_1 u_2 = v_1 v_2$;

(ii) $u_1 = v_1$ and either $u_2 \in |C_e(v_2)$ or $v_2 \in$ $\in |C_e(u_2)$;

(iii) $u_2 = v_2$ and either $u_1 \in |C_e(v_1)$ or $v_1 \in$ $\in |C_e(u_1)$ can take place.

Let e be an arbitrary Γ-equation. We call an e-proof $\ulcorner t_1, \dots, t_m \urcorner$ regular if either $t_i \in LC_e(t_{i+1})$ for all leaps i in $\ulcorner t_1, \dots, t_m \urcorner$ or $t_{i+1} \in LC_e(t_i)$ for all leaps i in $\ulcorner t_1, \dots, t_m \urcorner$. Evidently, if an e-proof has at most one leap, then it is regular.

Lemma 1. Let $a, b \in W_\Gamma$ and $e_1 \vdash \langle a, b \rangle$. Then there exists a regular e_1-proof of b from a.

Proof. Let $\ulcorner u_1, \dots, u_m \urcorner$ be an e_1-proof of b from a with a minimal number of leaps. Suppose that it is not regular. Evidently, it has two leaps i and j $(i < j)$ such that there is no leap greater than i and smaller than j (we say that i and j are two neighbouring leaps) and such that either

$u_i = (\alpha\alpha . \alpha)\beta \,\&\, u_{i+1} = \alpha\beta \,\&\, u_j = \gamma\delta \,\&\, u_{j+1} = (\gamma\gamma . \gamma)\delta$

or

$u_i = \alpha\beta \,\&\, u_{i+1} = (\alpha\alpha . \alpha)\beta \,\&\, u_j = (\gamma\gamma . \gamma)\delta \,\&\, u_{j+1} = \gamma\delta$

for some $\alpha, \beta, \gamma, \delta \in W_\Gamma$. If $i + 1 = j$, then $\alpha =$ $= \gamma$ and $\beta = \delta$, so that $\ulcorner u_1, \dots, u_i, u_{i+3}, \dots, u_m \urcorner$

is an e_1 -proof of b from a which has a smaller number of leaps than $\ulcorner u_1, \dots, u_n \urcorner$, a contradiction. Let $i + 1 < j$. In the first case

$$\ulcorner u_1, \dots, u_i, ((\overleftarrow{u}_{i+2} . \alpha)\alpha)\beta, \dots, ((\overleftarrow{u}_j . \alpha)\alpha)\beta ,$$
$$((\overleftarrow{u}_j . \overleftarrow{u}_{i+2})\alpha)\beta, \dots, ((\overleftarrow{u}_j . \overleftarrow{u}_j)\alpha)\beta ,$$
$$((\overleftarrow{u}_j . \overleftarrow{u}_j)\overleftarrow{u}_{i+2})\beta, \dots, ((\overleftarrow{u}_j . \overleftarrow{u}_j)\overleftarrow{u}_j)\beta, ((\overleftarrow{u}_j . \overleftarrow{u}_j)\overleftarrow{u}_j)\overrightarrow{u}_{i+2}, \dots,$$
$$((\overleftarrow{u}_j . \overleftarrow{u}_j)\overleftarrow{u}_j)\overrightarrow{u}_j, u_{j+2}, \dots, u_n \urcorner$$

and in the second case

$$\ulcorner u_1, \dots, u_i, \overrightarrow{\overrightarrow{u}}_{i+2} . \overrightarrow{u}_{i+2}, \dots, \overrightarrow{\overrightarrow{u}}_j . \overrightarrow{u}_j, u_{j+2}, \dots, u_n \urcorner$$

is an e_1 -proof of b from a and it has a smaller number of leaps than $\ulcorner u_1, \dots, u_n \urcorner$, a contradiction.

Lemma 2. Let $a_1 , a_2, b_1, b_2 \in W_r$. Then $e_1 \vdash \langle a_1 a_2 , b_1 b_2 \rangle$ if and only if $e_1 \vdash \langle a_2, b_2 \rangle$ and one of the following three cases takes place:

(i) $e_1 \vdash \langle a_1 , b_1 \rangle$;

(ii) $e_1 \vdash \langle a_1 , \sigma_n (b_1) \rangle$ for some $n \geq 1$;

(iii) $e_1 \vdash \langle b_1 , \sigma_n (a_1) \rangle$ for some $n \geq 1$.

Proof follows easily from Lemma 1.

Lemma 3. For every $t \in W_r$ denote by φ_t the endomorphism of W_r assigning t to every variable. Let $x \in X$, $a \in W_r$ and $w \in T_r (x)$; let $w \neq x$. Then $\{e_1, e_2\} \vdash \langle a, \varphi_a (w) \rangle$ does not hold.

Proof by the induction on a . Everything is evident if $a \in X$. Let $a \notin X$ and suppose $\{e_1, e_2\} \vdash \langle a, \varphi_a (w) \rangle$. Evidently, there exists a finite sequence $w_1, \dots, w_n$ such that $w_1 = w$, $w_n = x$ and $w_{i+1} = \overrightarrow{w}_i$ for every

$i = 1, \dots, n-1$. We have evidently $\{e_1, e_2\} \vdash \langle \vec{a}, \varphi_a(w_2) \rangle$; from this $\{e_1, e_2\} \vdash \langle \vec{\vec{a}}, \varphi_a(w_3) \rangle$; etc; finally, $\{e_1, e_2\} \vdash \langle b, \varphi_a(w_n) \rangle = \langle b, a \rangle$ for some $b \in S(\vec{a})$, so that $\{e_1, e_2\} \vdash \langle b, \varphi_b(w) \rangle$, a contradiction with the induction assumption.

<u>Lemma 4</u>. Let $a, b \in W_\Gamma$ and $e_2 \vdash \langle a, b \rangle$. Then there exists an e_2-proof of b from a which has at most one leap.

<u>Proof</u>. Let $\ulcorner u_1, \dots, u_m \urcorner$ be an e_2-proof of b from a with a minimal number of leaps. Suppose that it has at least two leaps. Then it has two neighbouring leaps i and j $(i < j)$. Four cases are possible:

(1) There exist $\alpha, \beta \in W_\Gamma$ such that $u_i = (\alpha \cdot \alpha\alpha)\alpha \;\&\; u_{i+1} = \alpha\alpha \;\&\; u_j = (\beta \cdot \beta\beta)\beta \;\&\; u_{j+1} = \beta\beta$; then $e_2 \vdash \langle \alpha, \beta \cdot \beta\beta \rangle$ and $e_2 \vdash \langle \alpha, \beta \rangle$, so that $e_2 \vdash \langle \beta, \beta \cdot \beta\beta \rangle$, a contradiction with Lemma 3.

(2) There exist $\alpha, \beta \in W_\Gamma$ such that $u_i = \alpha\alpha \;\&\; u_{i+1} = (\alpha \cdot \alpha\alpha)\alpha \;\&\; u_j = \beta\beta \;\&\; u_{j+1} = (\beta \cdot \beta\beta)\beta$; then $e_2 \vdash \langle \alpha, \alpha \cdot \alpha\alpha \rangle$, a contradiction.

(3) and (4) The remaining two cases give a contradiction similarly as in the proof of Lemma 1.

<u>Lemma 5</u>. Let $a_1, a_2, b_1, b_2 \in W_\Gamma$. Then $e_2 \vdash \langle a_1 a_2, b_1 b_2 \rangle$ if and only if $e_2 \vdash \langle a_2, b_2 \rangle$ and one of the following three cases takes place:

(i) $e_2 \vdash \langle a_1, b_1 \rangle$;

(ii) $e_2 \vdash \langle a_1, a_2 \rangle$ and $e_2 \vdash \langle b_1, a_1 \cdot a_1 a_1 \rangle$;

(iii) $e_2 \vdash \langle b_1, b_2 \rangle$ and $e_2 \vdash \langle a_1, b_1 \cdot b_1 b_1 \rangle$.

Proof follows easily from Lemma 4.

Lemma 6. Let $\alpha, \beta \in W_\Gamma$. Then neither $\{e_1, e_2\} \vdash \langle \alpha\alpha . \alpha, \beta . \beta\beta \rangle$ nor $\{e_1, e_2\} \vdash \langle \alpha\alpha . \alpha, \beta\beta \rangle$ takes place.

Proof. Suppose on the contrary that there exist elements $\alpha, \beta \in W_\Gamma$ and an $\{e_1, e_2\}$-proof $\ulcorner u_1, \ldots, u_m \urcorner$ such that the following holds: $u_1 = \alpha\alpha . \alpha$; either $u_m = \beta . \beta\beta$ or $u_m = \beta\beta$; whenever $\gamma, \delta \in W_\Gamma$ and $\ulcorner v_1, \ldots, v_n \urcorner$ is an $\{e_1, e_2\}$-proof of either $\delta . \delta\delta$ or $\delta\delta$ from $\gamma\gamma . \gamma$, then $n \leq m$. This $\ulcorner u_1, \ldots, u_m \urcorner$ has leaps, for if it had not, then in case $u_m = \beta . \beta\beta$ we would have $\{e_1, e_2\} \vdash \langle \alpha\alpha, \beta \rangle$ and $\{e_1, e_2\} \vdash \langle \alpha, \beta\beta \rangle$, so that $\{e_1, e_2\} \vdash \langle \alpha, \alpha\alpha . \alpha\alpha \rangle$; and in case $u_m = \beta\beta$ we would have $\{e_1, e_2\} \vdash \langle \alpha\alpha, \beta \rangle$ and $\{e_1, e_2\} \vdash \langle \alpha, \beta \rangle$, so that $\{e_1, e_2\} \vdash \langle \alpha, \alpha\alpha \rangle$, a contradiction with Lemma 3. Let i be the first leap in $\ulcorner u_1, \ldots, u_m \urcorner$.

If $u_i = (\varkappa\varkappa . \varkappa)\lambda$ & $u_{i+1} = \varkappa\lambda$ for some $\varkappa, \lambda \in W_\Gamma$, then $\ulcorner \bar{u}_i, \bar{u}_{i-1}, \ldots, \bar{u}_1 \urcorner$ is an $\{e_1, e_2\}$-proof of $\alpha\alpha$ from $\varkappa\varkappa . \varkappa$, and $i < m$ gives a contradiction.

If $u_i = (\varkappa . \varkappa\varkappa)\varkappa$ & $u_{i+1} = \varkappa\varkappa$, then $\{e_1, e_2\} \vdash \langle \alpha\alpha, \varkappa . \varkappa\varkappa \rangle$ and $\{e_1, e_2\} \vdash \langle \alpha, \varkappa \rangle$, so that $\{e_1, e_2\} \vdash \langle \alpha, \alpha\alpha \rangle$, a contradiction with Lemma 3.

If $u_i = \varkappa\varkappa$ & $u_{i+1} = (\varkappa . \varkappa\varkappa)\varkappa$, then $\{e_1, e_2\} \vdash \langle \alpha, \alpha\alpha \rangle$, a contradiction.

Let us call a leap ℓ in $\ulcorner u_1, \ldots, u_m \urcorner$ a $*$-leap if there exist $\varkappa, \lambda \in W_\Gamma$ such that $u_\ell = \varkappa\lambda$ & $u_{\ell+1} =$

$= (rr.r)s$. We have proved that i is a $*$-leap. Suppose that every leap in $\ulcorner u_1, \ldots, u_m \urcorner$ is a $*$-leap. Then $\{e_1, e_2\} \vdash \langle \beta, \sigma_m(\alpha\alpha) \rangle$ for some $m \geq 1$; in case $u_m = \beta . \beta\beta$ we have further $\{e_1, e_2\} \vdash \langle \alpha, \beta\beta \rangle$, so that $\{e_1, e_2\} \vdash \langle \alpha, \sigma_m(\alpha\alpha) . \sigma_m(\alpha\alpha) \rangle$, a contradiction; in case $u_m = \beta\beta$ we have $\{e_1, e_2\} \vdash \langle \alpha, \beta \rangle$, so that $\{e_1, e_2\} \vdash \langle \alpha, \sigma_m(\alpha\alpha) \rangle$, a contradiction again. This proves that $\ulcorner u_1, \ldots, u_m \urcorner$ has two neighbouring leaps j and k ($j < k$) such that k is not a $*$-leap and j is a $*$-leap. There exist a, $b \in W_\Gamma$ such that $u_j = ab \;\&\; u_{j+1} = (aa.a)b$.

Suppose $u_k = (cc.c)d \;\&\; u_{k+1} = cd$ for some c, $d \in W_\Gamma$. Then

$$\ulcorner u_1, \ldots, u_j, \overrightarrow{\overleftarrow{u}}_{j+2} . \vec{u}_{j+2}, \ldots, \overrightarrow{\overleftarrow{u}}_k . \vec{u}_k, u_{k+2}, \ldots, u_m \urcorner$$

is an $\{e_1, e_2\}$-proof, a contradiction with the minimal property of $\ulcorner u_1, \ldots, u_m \urcorner$.

Suppose $u_k = (c.cc)c \;\&\; u_{k+1} = cc$. Then $\ulcorner \overleftarrow{u}_{j+1}, \ldots, \overleftarrow{u}_k \urcorner$ is an $\{e_1, e_2\}$-proof of $c.cc$ from $aa.a$, a contradiction with the minimal property of $\ulcorner u_1, \ldots, u_m \urcorner$.

The case $u_k = cc \;\&\; u_{k+1} = (c.cc)c$ remains. $\ulcorner u_1, \ldots, u_k \urcorner$ is an $\{e_1, e_2\}$-proof of cc from $\alpha\alpha.\alpha$, again a contradiction with the minimal property of $\ulcorner u_1, \ldots, u_m \urcorner$.

Lemma 7. $Cn(e_1) \vee_\Gamma Cn(e_2) = \iota_\Gamma$.

Proof. Let us prove the following assertion by induction on a: whenever $a, b \in W_\Gamma$, $e_1 \vdash \langle a, b \rangle$ and

$e_2 \vdash \langle a, b \rangle$, then $a = b$. This is evident if $a \in$ $\in X$. Let $a = a_1 a_2$.
Evidently, $b \notin X$; put $b = b_1 b_2$. We get $a_2 = b_2$ easily from the induction assumption, so that it is enough to prove $a_1 = b_1$.

Let $e_1 \vdash \langle a_1, b_1 \rangle$. By Lemma 5, the following three cases are the only possible ones:

(1) $e_2 \vdash \langle a_1, b_1 \rangle$. Then we get $a_1 = b_1$ from the induction assumption.

(2) $e_2 \vdash \langle a_1, a_2 \rangle \,\&\, e_2 \vdash \langle b_1, a_1 . a_1 a_1 \rangle$. As $\{e_1, e_2\} \vdash \langle a_1, a_1 . a_1 a_1 \rangle$, we get a contradiction with Lemma 3.

(3) $e_2 \vdash \langle b_1, b_2 \rangle \,\&\, e_2 \vdash \langle a_1, b_1 . b_1 b_1 \rangle$. Again, $\{e_1, e_2\} \vdash \langle b_1, b_1 . b_1 b_1 \rangle$, a contradiction.

Let $e_1 \vdash \langle a_1, \sigma_m(b_1) \rangle$ for some $m \geq 1$. (1), (2) and (3) are again the only possible cases. In cases (1) and (2) we get a contradiction with Lemma 3. In case (3) we get a contradiction with Lemma 6 and the definition of σ_m .

By Lemma 5, the case $e_1 \vdash \langle b_1, \sigma_m(a_1) \rangle$ remains. This case is similar to $e_1 \vdash \langle a_1, \sigma_m(b_1) \rangle$.

<u>Lemma 8</u>. If $a \in W_r$, then $e^1 \vdash \langle a, a a \rangle$ does not hold.

<u>Proof</u> by induction on a . It is evident if $a \in X$. Let $a = a_1 a_2$ and suppose $e^1 \vdash \langle a, a a \rangle$. Evidently, $e^1 \vdash \langle a_2, a \rangle$, so that $e^1 \vdash \langle a_2, a_2 a_2 \rangle$ which contradicts to the induction assumption.

<u>Lemma 9</u>. Let $a, b \in W_r$ and $e^1 \vdash \langle a, b \rangle$. Then

there exists an e^1 -proof of $\mathscr{b}$ from $\mathscr{a}$ which has at most one leap.

Proof. Let $\ulcorner u_1, \dots, u_m \urcorner$ be an e^1 -proof of $\mathscr{b}$ from $\mathscr{a}$ with a minimal number of leaps. Suppose that it has at least two leaps, so that it has two neighbouring leaps i and j $(i < j)$. There are four cases:

(1) $u_i = \alpha\alpha \;\&\; u_{i+1} = \alpha\alpha.\alpha \;\&\; u_j = \beta\beta \;\&\; u_{j+1} = \beta\beta.\beta$ for some $\alpha, \beta \in W_T$. Then $e^1 \vdash \langle \alpha\alpha, \beta \rangle$ and $e^1 \vdash \langle \alpha, \beta \rangle$, so that $e^1 \vdash \langle \alpha, \alpha\alpha \rangle$, a contradiction with Lemma 8.

(2) $u_i = \alpha\alpha.\alpha \;\&\; u_{i+1} = \alpha\alpha \;\&\; u_j = \beta\beta.\beta \;\&\; u_{j+1} = \beta\beta$. We can get a contradiction similarly as in the preceding case.

(3) $u_i = \alpha\alpha \;\&\; u_{i+1} = \alpha\alpha.\alpha \;\&\; u_j = \beta\beta.\beta \;\&\; u_{j+1} = \beta\beta$.
Then
$\ulcorner u_1, \dots, u_i, \vec{u}_{i+2}.\alpha, \dots, \vec{u}_j.\alpha, \vec{u}_j.\vec{u}_{i+2}, \dots, \vec{u}_j.\vec{u}_j, u_{j+2}, \dots, u_n \urcorner$
is an e^1 -proof of $\mathscr{b}$ from $\mathscr{a}$ which has a smaller number of leaps than $\ulcorner u_1, \dots, u_m \urcorner$, a contradiction.

(4) $u_i = \alpha\alpha.\alpha \;\&\; u_{i+1} = \alpha\alpha \;\&\; u_j = \beta\beta \;\&\; u_{j+1} = \beta\beta.\beta$.
Then
$\ulcorner u_1, \dots, u_i, u_{i+2}.\alpha, \dots, u_j.\alpha, u_j.\overrightarrow{u_{i+2}}, \dots, u_j.\vec{u}_j, u_{j+2}, \dots, u_n \urcorner$
is an e^1 -proof of $\mathscr{b}$ from $\mathscr{a}$ which has a smaller number of leaps, a contradiction again.

Lemma 10. Let $a_1, a_2, \mathscr{b}_1, \mathscr{b}_2 \in W_T$. Then $e^1 \vdash \langle a_1 a_2, \mathscr{b}_1 \mathscr{b}_2 \rangle$ if and only if $e^1 \vdash \langle a_2, \mathscr{b}_2 \rangle$ and one of the following three cases takes place:

(i) $e^1 \vdash \langle a_1, \mathscr{b}_1 \rangle$;

(ii) $e^1 \vdash \langle b_1, b_2 \rangle$ and $e^1 \vdash \langle a_1, b_1 b_1 \rangle$;

(iii) $e^1 \vdash \langle a_1, a_2 \rangle$ and $e^1 \vdash \langle b_1, a_1 a_1 \rangle$.

Proof follows easily from Lemma 9.

Lemma 11. Let $a_1, a_2, b_1, b_2 \in W_\Gamma$. Then $e^2 \vdash \langle a_1 a_2, b_1 b_2 \rangle$ if and only if $e^2 \vdash \langle a_1, b_1 \rangle$ and one of the following three cases takes place:

(i) $e^2 \vdash \langle a_2, b_2 \rangle$;

(ii) $e^2 \vdash \langle b_1, b_2 \rangle$ and $e^2 \vdash \langle a_2, b_2 b_2 \rangle$;

(iii) $e^2 \vdash \langle a_1, a_2 \rangle$ and $e^2 \vdash \langle b_2, a_2 a_2 \rangle$.

Proof is similar to that of Lemma 10.

Lemma 12. Let $a, b \in W_\Gamma$. If $\{e^1, e^2\} \vdash \langle aa, bb \rangle$, then $\{e^1, e^2\} \vdash \langle a, b \rangle$, too.

Proof. Suppose that it is not true. There exists an $\{e^1, e^2\}$-proof $\ulcorner u_1, \dots, u_m \urcorner$ such that the following holds: there exist $\alpha, \beta \in W_\Gamma$ satisfying $u_1 = \alpha\alpha$ and $u_m = \beta\beta$ and not satisfying $\{e^1, e^2\} \vdash \langle \alpha, \beta \rangle$; whenever $\ulcorner v_1, \dots, v_n \urcorner$ is an $\{e^1, e^2\}$-proof with a similar property, then $n \leq m$. Choose such a minimal $\ulcorner u_1, \dots, u_m \urcorner$ and put $u_1 = aa$ and $u_2 = bb$. Suppose $u_i = cc$ for some i such that $2 \leq i \leq m - 1$. As $\ulcorner u_1, \dots, u_i \urcorner$ is an $\{e^1, e^2\}$-proof of cc from aa and $i < m$, we have $\{e^1, e^2\} \vdash \langle a, c \rangle$; as $\ulcorner u_i, \dots, u_m \urcorner$ is an $\{e^1, e^2\}$-proof of bb from cc and $m - i + 1 < m$, we have $\{e^1, e^2\} \vdash \langle b, c \rangle$. Consequently, $\{e^1, e^2\} \vdash \langle a, b \rangle$, a contradiction. From this we infer that no numbers other than 1 and $m - 1$ can be leaps in $\ulcorner u_1, \dots, u_m \urcorner$. If $\ulcorner u_1, \dots, u_m \urcorner$ had at most one leap, then either $\ulcorner \overleftarrow{u}_1, \dots, \overleftarrow{u}_m \urcorner$ or $\ulcorner \overrightarrow{u}_1, \dots, \overrightarrow{u}_m \urcorner$ would be an $\{e^1, e^2\}$-proof of b from a; hence, the numbers 1 and $m-1$ are leaps. We have either $u_2 = aa.a$ or $u_2 = a.aa$. It is sufficient to consider the case $u_2 = aa.a$. If it were $u_{m-1} = bb.b$, then $\ulcorner \overrightarrow{u}_1, \dots, \overrightarrow{u}_m \urcorner$ would be an $\{e^1, e^2\}$-proof of b from a. We get $u_{m-1} = b.bb$. Evidently, $\ulcorner \overrightarrow{u}_1, \dots, \overrightarrow{u}_{m-1} \urcorner$ is an $\{e^1, e^2\}$-proof of bb from a and $\ulcorner \overleftarrow{u}_2, \dots, \overleftarrow{u}_m \urcorner$ is an $\{e^1, e^2\}$-proof of b from aa. As $\{e^1, e^2\} \vdash \langle bb, a, a \rangle$, we get $\{e^1, e^2\} \vdash \langle a, b \rangle$, a contradiction.

Lemma 13. If $a \in W_\Gamma$, then $\{e^1, e^2\} \vdash \langle a, aa \rangle$ does not hold.

Proof by induction on a. It is evident if $a \in X$. Let $a = a_1 a_2$ and suppose $\{e^1, e^2\} \vdash \langle a, aa \rangle$. Let $\ulcorner u_1, \dots, u_m \urcorner$ be an arbitrary $\{e^1, e^2\}$-proof of a from aa.

Suppose that $\ulcorner u_1, \dots, u_m \urcorner$ has a leap. Denote by k its last leap. If it were $u_{k+1} = cc$ for some $c \in W_\Gamma$, then we would get $\{e^1, e^2\} \vdash \langle a_1, c \rangle$; as $\{e^1, e^2\} \vdash \langle aa, cc \rangle$, Lemma 12 gives $\{e^1, e^2\} \vdash \langle a, c \rangle$; hence, $\{e^1, e^2\} \vdash \langle a_1, a \rangle$, so that $\{e^1, e^2\} \vdash \langle a_1, a_1 a_1 \rangle$, a contradiction with the induction hypothesis. This proves $u_k = cc$ for some c and either $u_{k+1} = cc.c$ or $u_{k+1} = c.cc$. Again, from $\{e^1, e^2\} \vdash \langle aa, cc \rangle$ follows by Lemma 12 $\{e^1, e^2\} \vdash \langle a, c \rangle$. In case $u_{k+1} = cc.c$ we have $\{e^1, e^2\} \vdash \langle c, a_2 \rangle$, so that $\{e^1, e^2\} \vdash \langle a, a_2 \rangle$ and consequently $\{e^1, e^2\} \vdash \langle a_2, a_2 a_2 \rangle$, a contradiction with the induction hypothesis; in case $u_{k+1} = c.cc$ similarly $\{e^1, e^2\} \vdash \langle a_1, a_1 a_1 \rangle$, a contradiction again.

We have proved that $\ulcorner u_1, \dots, u_m \urcorner$ has no leaps. $\ulcorner \overleftarrow{u}_1, \dots, \overleftarrow{u}_m \urcorner$ is an $\{e^1, e^2\}$-proof of a_1 from a, so that $\{e^1, e^2\} \vdash \langle a_1, a_1 a_1 \rangle$, a contradiction with the induction hypothesis.

Lemma 14. $Cn(e^1) \vee_r Cn(e^2) = \iota_r$.

Proof. We shall prove by induction on a the following: whenever $e^1 \vdash \langle a, b \rangle$ and $e^2 \vdash \langle a, b \rangle$, then $a = b$. This is evident if $a \in X$. Let $a = a_1 a_2$, $e^1 \vdash \langle a, b \rangle$, $e^2 \vdash \langle a, b \rangle$ and $a \neq b$. Evidently, $b \notin X$; put $b = b_1 b_2$. We have $e^1 \vdash \langle a_2, b_2 \rangle$ and $e^2 \vdash \langle a_1, b_1 \rangle$; it is sufficient to prove $e^1 \vdash \langle a_1, b_1 \rangle$ and $e^2 \vdash \langle a_2, b_2 \rangle$. Suppose on the contrary e.g. that $e^1 \vdash \langle a_1, b_1 \rangle$ does not hold. We have either $e^1 \vdash \langle b_1, b_2 \rangle$ & $e^1 \vdash \langle a_1, b_1 b_1 \rangle$ or $e^1 \vdash \langle a_1, a_2 \rangle$ & & $e^1 \vdash \langle b_1, a_1 a_1 \rangle$ by Lemma 10. Evidently, $\{e^1, e^2\} \vdash \langle a_1, a_1 a_1 \rangle$ in both cases, a contradiction with Lemma 13.

Lemma 15. Let x and y be two different variables. Then every minimal $\langle xx.y, x.yx \rangle$-proof is regular.

Proof. Put $e = \langle xx.y, x.yx \rangle$. We shall prove by induction on m that every minimal e-proof $\ulcorner u_1, \dots, u_m \urcorner$ is regular. This is evident if $m = 1$. Let $m > 1$. Suppose that $\ulcorner u_1, \dots, u_m \urcorner$ is not regular, so that it has two neighbouring leaps i and j $(i < j)$ such that one of the following two cases takes place:

(1) $u_i = a.ba$ & $u_{i+1} = aa.b$ & $u_j = cc.d$ & $u_{j+1} = c.dc$ for some $a, b, c, d \in W_r$. We have $e \vdash \langle aa, cc \rangle$, so that $\ell(aa) = \ell(cc)$ and thus $\ell(a) = \ell(c)$. The e-proof

$\ulcorner \overleftarrow{u}_{i+1}, \dots, \overleftarrow{u}_j \urcorner$ of cc from aa is minimal if we leave out its members $\overleftarrow{u}_k$ such that $\overleftarrow{u}_k = \overleftarrow{u}_{k-1}$; by the induction assumption it follows easily from $\ell(a) = \ell(c)$ that $\ulcorner \overleftarrow{u}_{i+1}, \dots, \overleftarrow{u}_j \urcorner$ has no leaps. Consequently,

$$\ulcorner u_1, \dots, u_i, \overset{\leftleftarrows}{u}_{i+2} . (\overrightarrow{u}_{i+2} . \overset{\rightrightarrows}{u}_{i+2}), \dots, \overset{\rightrightarrows}{u}_j . (\overrightarrow{u}_j . \overset{\rightrightarrows}{u}_j), u_{j+2}, \dots, u_n \urcorner$$

is an e -proof of u_n from u_1 , a contradiction with the minimality of $\ulcorner u_1, \dots, u_n \urcorner$.

(2) $u_i = aa . b \mathbin{\&} u_{i+1} = a . ba \mathbin{\&} u_j = c . dc \mathbin{\&} u_{j+1} = cc . d$ for some $a, b, c, d \in W_\Gamma$. We have $e \vdash \langle a, c \rangle$ and $e \vdash \langle ba, dc \rangle$, so that $\ell(a) = \ell(c)$ and $\ell(ba) = \ell(dc)$; we infer $\ell(b) = \ell(d)$. Similarly as in the previous case, $\ulcorner \overrightarrow{u}_{i+1}, \dots, \overrightarrow{u}_j \urcorner$ has no leaps and

$$\ulcorner u_1, \dots, u_i, (\overleftarrow{u}_{i+2} . \overset{\rightrightarrows}{u}_{i+2}) . \overset{\rightleftarrows}{u}_{i+2}, \dots, (\overleftarrow{u}_j . \overset{\rightrightarrows}{u}_j) . \overset{\rightleftarrows}{u}_j, u_{j+2}, \dots, u_n \urcorner$$

is a shorter proof of u_n from u_1 , a contradiction.

Lemma 16. Let x and y be two different variables. Then

$$Cn(\langle xx . y, x . yx \rangle) \vee_\Gamma Cn(\langle x . (xx . x), (xx . x) . x \rangle) = \iota_\Gamma .$$

Proof. Put $e = \langle xx . y, x . yx \rangle$ and $\bar{e} = \langle x . (xx . x), (xx . x) . x \rangle$. Let $a, b \in W_\Gamma$, $e \vdash \langle a, b \rangle$ and $\bar{e} \vdash \langle a, b \rangle$. Suppose that a minimal e -proof of b from a has leaps. Using Lemma 15, there exists a natural number $n \geq 1$ such that either $\ell(\overleftarrow{a}) = 2^n . \ell(\overleftarrow{b})$ or $\ell(\overleftarrow{b}) = 2^n . \ell(\overleftarrow{a})$. By Lemma 1 of [6], a minimal $\bar{e}$ -proof of b from a has at most one leap. If it has a leap, we have either $\ell(\overleftarrow{a}) = 3 . \ell(\overleftarrow{b})$ or $\ell(\overleftarrow{b}) = 3 . \ell(\overleftarrow{a})$; if it has not, we have $\ell(\overleftarrow{a}) = \ell(\overleftarrow{b})$. This gives a contradiction in each case, as neither $2^n = 3$ nor $2^n = \frac{1}{3}$ nor $2^n = 1$.

We have proved that a minimal e-proof of b from a has no leaps. This implies $\ell(\overleftarrow{a}) = \ell(\overleftarrow{b})$ and a minimal $\bar{e}$-proof of b from a has no leaps, too. If we had proved the equality by induction on a, we should get $\overleftarrow{a} = \overleftarrow{b}$ and $\overrightarrow{a} = \overrightarrow{b}$, so that $a = b$.

<u>Lemma 17</u>. Let x and y be two different variables; put $e = \langle x.yx, xy.x \rangle$. Then every minimal e-proof has at most one leap.

<u>Proof</u>. We shall prove by induction on n that every minimal e-proof $\ulcorner u_1, \dots, u_n \urcorner$ has at most one leap. This is evident if $n = 1$. Let $n > 1$ and suppose that a minimal e-proof $\ulcorner u_1, \dots, u_n \urcorner$ has at least two leaps. It has two neighbouring leaps i and j $(i < j)$; one of the following four cases takes place:

(1) $u_i = a.ba \;\&\; u_{i+1} = ab.a \;\&\; u_j = c.dc \;\&\; u_{j+1} = cd.c$ for some $a, b, c, d \in W_\Gamma$. We have $e \vdash \langle ab, c \rangle$ and $e \vdash \langle a, dc \rangle$, so that $\ell(ab) = \ell(c)$ and $\ell(a) = \ell(dc)$ and consequently $\ell(ab) < \ell(a)$, which is impossible.

(2) $u_i = ab.a \;\&\; u_{i+1} = a.ba \;\&\; u_j = cd.c \;\&\; u_{j+1} = c.dc$; a contradiction can be derived similarly.

(3) $u_i = a.ba \;\&\; u_{i+1} = ab.a \;\&\; u_j = cd.c \;\&\; u_{j+1} = c.dc$. We have $\ell(ab) = \ell(cd)$ and $\ell(a) = \ell(c)$ and consequently $\ell(b) = \ell(d)$, too. By the induction hypothesis, this implies that $\ulcorner \overleftarrow{u}_{i+1}, \dots, \overleftarrow{u}_j \urcorner$ has no leaps, so that

$\ulcorner u_1, \dots, u_i, \overleftarrow{u}_{i+2}.(\overrightarrow{u}_{i+2}.\overrightarrow{u}_{i+2}), \dots, \overleftarrow{u}_j.(\overrightarrow{u}_j.\overrightarrow{u}_j), u_{j+2}, \dots, u_n \urcorner$

is a shorter e-proof of u_n from u_1, a contradiction.

(4) $u_i = ab.a \;\&\; u_{i+1} = a.ba \;\&\; u_j = c.dc \;\&\; u_{j+1} = cd.c$; we can get a contradiction similarly.

Lemma 18. Let x and y be two different variables. Then

$Cn(\langle x.yx, xy.x\rangle) \vee_r Cn(\langle x.(xx.xx), (xx.xx).x\rangle) = \iota_r$.

Proof. Put $e = \langle x.yx, xy.x\rangle$ and $\bar{e} = \langle x.(xx.xx), (xx.xx).x\rangle$. We prove the following by induction on a: whenever $e \vdash \langle a, b\rangle$ and $\bar{e} \vdash \langle a, b\rangle$, then $a = b$. This is evident if $a \in X$. Let $a = a_1 a_2$, $e \vdash \langle a, b\rangle$ and $\bar{e} \vdash \langle a, b\rangle$. Evidently, $b \notin X$; put $b = b_1 b_2$. Let $\ulcorner u_1, \dots, u_n \urcorner$ be a minimal $\bar{e}$-proof of b from a. By Lemma 1 of [6] it has at most one leap.

Suppose that $\ulcorner u_1, \dots, u_n \urcorner$ has exactly one leap i. It is sufficient to consider only the case $u_i = \alpha.(\alpha\alpha.\alpha\alpha)$ & $u_{i+1} = (\alpha\alpha.\alpha\alpha).\alpha$ for some $\alpha \in W_r$. As $\ell(\alpha\alpha) = \ell(\alpha\alpha)$, the $\bar{e}$-proof $\ulcorner \overleftarrow{u}_{i+1}, \dots, \overleftarrow{u}_n \urcorner$ has no leaps. Hence, $\ell(b_1) = 4.\ell(a_1)$, $b_1 \notin X$ and $\ell(\overleftarrow{b_1}) = 2.\ell(a_1) = \ell(\overrightarrow{b_1})$. Let $\ulcorner v_1, \dots, v_m \urcorner$ be a minimal e-proof of b from a. As $\ell(a_1) < \ell(b_1)$, $\ulcorner v_1, \dots, v_m \urcorner$ has leaps; by Lemma 17, it has exactly one leap j; evidently, there exist $\beta, \gamma \in W_r$ such that $v_j = \beta.\gamma\beta$ & $v_{j+1} = \beta\gamma.\beta$. As $\ulcorner \overleftarrow{v}_{j+1}, \dots, \overleftarrow{v}_m \urcorner$ is (after leaving its members $\overleftarrow{v}_\ell$ such that $\overleftarrow{v}_\ell = \overleftarrow{v}_{\ell-1}$) a minimal e-proof, it has at most one leap; as $\ell(\beta) = \ell(a_1)$ and $\ell(\overleftarrow{b_1}) = 2.\ell(a_1)$, it has exactly one leap k and there exist ε and δ such that $\overleftarrow{v}_k = \delta.\varepsilon\delta$ & $\overleftarrow{v}_{k+1} = \delta\varepsilon.\delta$. We get $\ell(\overrightarrow{b_1}) = \ell(\delta) = \ell(\beta) = \ell(a_1)$, a contradiction with $\ell(\overrightarrow{b_1}) = 2.\ell(a_1)$.

We have proved that $\ulcorner u_1, \dots, u_n \urcorner$ has no leaps and consequently $\bar{e} \vdash \langle a_1, b_1\rangle$ and $\bar{e} \vdash \langle a_2, b_2\rangle$. As

$\ell(a_1) = \ell(b_1)$, a minimal e-proof of b from a has no leaps, too, so that $e \vdash \langle a_1, b_1 \rangle$ and $e \vdash \langle a_2, b_2 \rangle$. The induction assumption gives $a_1 = b_1$ and $a_2 = b_2$, so that $a = b$.

Lemma 19. Let x, y and z be three different variables; put $e = \langle ((x.xy)z)z, x(x(yz.z)) \rangle$. Then every minimal e-proof has at most one leap.

Proof. We prove by induction on n for every minimal e-proof $\ulcorner u_1, \dots, u_n \urcorner$ that it has at most one leap. The case $n = 1$ is evident; let $n > 1$ and suppose that $\ulcorner u_1, \dots, u_n \urcorner$ has at least two leaps. It has two neighbouring leaps i and j $(i < j)$; one of the following four cases takes place:

(1) $u_i = ((a.ab)c)c \;\&\; u_{i+1} = a(a(bc.c)) \;\&\; u_j = ((p.pq)r)r \;\&\; u_{j+1} = p(p(qr.r))$ for some $a, b, c, p, q, r \in W_\Gamma$. We have

$$\ell(a) = \ell((p.pq)r) > \ell(r) = \ell(a(bc.c)) > \ell(a),$$

a contradiction.

(2) $u_i = a(a(bc.c)) \;\&\; u_{i+1} = ((a.ab)c)c \;\&\; u_j = p(p(qr.r)) \;\&\; u_{j+1} = ((p.pq)r)r$; we get a contradiction similarly.

(3) $u_i = ((a.ab)c)c \;\&\; u_{i+1} = a(a(bc.c)) \;\&\; u_j = p(p(qr.r)) \;\&\; u_{j+1} = ((p.pq)r)r$. We have $\ell(a) = \ell(p)$ and $\ell(a(bc.c)) = \ell(p(qr.r))$, so that $\ell(bc.c) = \ell(qr.r)$. As the e-proof $\ulcorner \vec{u}_{i+1}, \dots, \vec{u}_j \urcorner$ is (after leaving out some members) minimal, it has no leaps by the induction assumption. Hence, $\ulcorner \vec{\vec{u}}_{i+1}, \dots, \vec{\vec{u}}_j \urcorner$ is an e-proof and it is minimal if we leave out some members; as $\ell(bc) > \ell(c)$ and $\ell(qr) > \ell(r)$, it has no leaps.

We get $\ell(bc) = \ell(qr)$ and $\ell(c) = \ell(r)$, so that $\ell(b) = \ell(q)$, too. Again, the e -proof $\ulcorner \overset{\leftrightarrows}{\vec{u}}_{i+1}, \ldots, \overset{\leftrightarrows}{\vec{u}}_{j} \urcorner$ has no leaps. Evidently,

$$\ulcorner u_1, \ldots, u_i, ((\overleftarrow{u}_{i+2} . (\overset{\leftrightarrows}{u}_{i+2} . \overset{\leftarrow}{\overset{\leftrightarrows}{u}}_{i+2})) . \overset{\rightarrow}{\overset{\leftrightarrows}{u}}_{i+2}) . \overset{\rightrightarrows}{\vec{u}}_{i+2}, \ldots$$

$$\ldots, ((\overleftarrow{u}_{j} . (\overset{\leftrightarrows}{u}_{j} . \overset{\leftarrow}{\overset{\leftrightarrows}{u}}_{j})) . \overset{\rightarrow}{\overset{\leftrightarrows}{u}}_{j}) . \overset{\rightrightarrows}{\vec{u}}_{j}, u_{j+2}, \ldots, u_n \urcorner$$

is a shorter e -proof of u_n from u_1, a contradiction.

(4) The last case is similar to the previous one.

<u>Lemma 20</u>. Let x, y and z be three different variables. Then

$$Cn(\langle((x.xy)z)z, x(x(yz.z))\rangle) \vee_r Cn(\langle x.xx, xx.x\rangle) = \iota_r .$$

<u>Proof</u>. Put $e = \langle((x.xy)z)z, x(x(yz.z))\rangle$ and $\bar{e} = \langle x.xx, xx.x\rangle$. We prove by induction on a : whenever $e \vdash \langle a, b\rangle$ and $\bar{e} \vdash \langle a, b\rangle$, then $a = b$. This is evident if $a \in X$. Let $a = a_1 a_2$, $e \vdash \langle a, b\rangle$ and $\bar{e} \vdash \langle a, b\rangle$. Evidently, $b \notin X$; put $b = b_1 b_2$. Let $\ulcorner u_1, \ldots, u_n \urcorner$ be a minimal $\bar{e}$ -proof of b from a. By Lemma 1 of [6], it has at most one leap. Suppose that it has exactly one leap i. It is sufficient to derive a contradiction in the case $u_i = \alpha.\alpha\alpha$ & $u_{i+1} = \alpha\alpha.\alpha$ for some $\alpha \in W_r$. We have $\ell(b_1) = 2.\ell(a_1)$. Hence, using Lemma 19, a minimal e -proof of b from a has exactly one leap, too, and for some $\beta, \gamma, \delta \in W_r$ $\ell(b_1) = \ell((\beta.\beta\gamma)\delta) > 2.\ell(\beta) = 2.\ell(a_1)$, a contradiction.

We have proved that $\ulcorner u_1, \ldots, u_n \urcorner$ has no leaps. We get $\bar{e} \vdash \langle a_1, b_1\rangle$ and $\bar{e} \vdash \langle a_2, b_2\rangle$, so that $\ell(a_1) = \ell(b_1)$ and a minimal e -proof of b from a has no leaps, too. This implies $e \vdash \langle a_1, b_1\rangle$ and

$e \vdash \langle a_2, b_2 \rangle$; by the induction assumption $a_1 = b_1$ and $a_2 = b_2$, so that $a = b$.

Lemma 21. Let x, y and z be three different variables; let e be any of the following eight equations: $\langle (xx.x)y, xy \rangle$; $\langle y(x.xx), yx \rangle$; $\langle xx.x, xx \rangle$; $\langle x.xx, xx \rangle$; $\langle xx.y, x.yx \rangle$; $\langle y.xx, xy.x \rangle$; $\langle x.yx, xy.x \rangle$; $\langle ((x.xy)z)z, x(x(yz.z)) \rangle$.
Then $Cn(e)$ is an upper semicomplement in $\mathcal{L}_\Gamma$.

Proof follows from Lemmas 7, 14, 16, 18 and 20 and their duals.

§ 2. The infimum of the set of all upper semicomplements in $\mathcal{L}_\Gamma$

Lemma 22. Let $x \in X$, $w \in W_\Gamma$ and $w \neq x$. Then $Cn(\langle x, w \rangle)$ is not an upper semicomplement in $\mathcal{L}_\Gamma$.

Proof. Suppose on the contrary that there exists a non-trivial equation $\langle a, b \rangle$ such that $Cn(\langle x, w \rangle \vee_\Gamma \vee_\Gamma Cn(\langle a, b \rangle) = \iota_\Gamma$. By Theorem 2 of [6], x is the only variable that is a subword of w; i.e. $w \in T_\Gamma(x)$. As $w \neq x$, there exist $u, v \in T_\Gamma(x)$ such that $w = uv$. For every two elements r, s of W_Γ define $r[s]$ by $r[s] = \varphi(r)$ where φ is the endomorphism of W_Γ , assigning s to each variable. The equation $e = \langle u[w[a]].v[w[b]], w[u[a].v[b]] \rangle$ is evidently non-trivial and we have both $\langle x, w \rangle \vdash e$ and $\langle a, b \rangle \vdash e$, a contradiction.

Lemma 23. Let x, y and z be three different variables. If $a, b \in W_\Gamma$, then

$\langle a, b \rangle \in Cn(\{\langle xx.y, xy \rangle, \langle xy, yx \rangle, \langle xy.z, x.yz \rangle\})$
if and only if $X \cap S(a) = X \cap S(b)$ and either $a = b$ or $a \notin X$ & $b \notin X$.

Proof is easy.

Theorem. The infimum in $\mathcal{L}_\Gamma$ of all upper semicomplements in $\mathcal{L}_\Gamma$ is just $Cn(\{\langle xx.y, xy \rangle, \langle xy, yx \rangle, \langle xy.z, x.yz \rangle\})$ (where x, y and z are three different variables).

Proof. Denote the infimum by E . (E is a fully invariant congruence relation of W_Γ .) By Lemma 21 we have $Cn(\{\langle xx.y, xy \rangle, \langle xy, yx \rangle, \langle xy.z, x.yz \rangle\}) \subseteq E$. The converse inclusion follows easily (some care is necessary) from Theorem 2 of [6] and Lemmas 22 and 23.

Denote by $\mathcal{G}$ the variety of all groupoids. We reformulate the theorem two times:

Corollary 1. For every groupoid A , the following two conditions are equivalent:

(i) $A \in \mathcal{U} \cap \mathcal{L}$ for every two proper subvarieties $\mathcal{U}, \mathcal{L}$ of $\mathcal{G}$ such that $\mathcal{G}$ is the only variety containing both $\mathcal{U}$ and $\mathcal{L}$;

(ii) A is a commutative semigroup satisfying $xx.y = xy$.

Corollary 2. Denote by E the set of all Γ -equations e such that $Cn(e)$ is an upper semicomplement in $\mathcal{L}_\Gamma$. Then
$Cn(E) = Cn(\{\langle xx.y, xy \rangle, \langle xy, yx \rangle, \langle xy.z, x.yz \rangle\})$.

Let L be an arbitrary lattice. An element $a \in L$

is called definable in L if there exists a formula φ of the first-order predicate calculus such that

(i) φ contains only logical symbols, variables and the two function symbols $\wedge$ and $\vee$;

(ii) φ has exactly one free variable;

(iii) a satisfies φ in L and no other element of L satisfies φ .

Any lattice L has at most countably many definable elements. The set of all definable elements of L is a sublattice of L . Every definable element is a fix-point of any automorphism of L .

If L has the greatest and the smallest element, then they are evidently both definable in L . A less trivial example is the supremum of all atoms in a complete atomic lattice L . Hence, the variety of all semigroups satisfying $xyzw = xzyw$ (see [3]) is a definable element in the lattice of all semigroup varieties. Unfortunately, the supremum of the set of all atoms in $\mathcal{L}_\Gamma$ is just the greatest element of $\mathcal{L}_\Gamma$ (see [1] or [5]). However, the theorem gives us

Corollary 3. $\mathcal{L}_\Gamma$ has definable elements different from the greatest and the smallest elements. $Cn(\{\langle xx.y, xy\rangle, \langle xy, yx\rangle, \langle xy.z, x.yz\rangle\})$ is a definable element.

The infimum of the set of all upper semicomplements is a definable element. It follows from Theorems 1 and 2 of [6] that if Δ is an arbitrary type containing at least one at least binary function symbol, then the infimum is a definable element in $\mathcal{L}_\Delta$, different from the extreme elements. It could be interesting to find this variety.

Problem. Find and describe other varieties of groupoids that are definable elements of $\mathcal{L}_\Gamma$. Are the important varieties (the variety of semigroups, commutative groupoids, commutative semigroups, idempotent groupoids, semilattices,...) definable in $\mathcal{L}_\Gamma$? Denote by Δ the type consisting of one binary, one unary and one nullary function symbol. Is the variety of groups definable in $\mathcal{L}_\Delta$?

The problem stated in [6] remains open.

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