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A CONTRIBUTION TO THE LINEAR HOMOGENEOUS DIOPHANTINE APPROXIMATIONS

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Let β be an irrational number and let $(b_0; b_1, b_2, \dots)$ its (simple) continued fraction expansion. A number $\beta' = (b'_0; b'_1, b'_2, \dots)$ will be called equivalent to β (notation $\beta' \sim \beta$ and $\beta' \not\sim \beta$ otherwise) if there exists integral numbers k and n_0 such that $b_n = b'_{k+n}$ for all natural numbers $n > n_0$. We shall use a standard notation for the period of a continued fraction; e.g. $(1; \overline{1, 2}) = (1; 1, 2, 1, 2, \dots) = \sqrt{3}$. For real $t \geq 1$ let

$$\psi_\beta(t) = \min_{\substack{p,q \text{ int.} \\ 0 < q \leq t}} |q\beta - p| .$$

It is well known that $0 < t\,\psi_\beta(t) < 1$ for every $t \geq 1$. Let

$$\mu(\beta) = \limsup_{t \to +\infty} t\,\psi_\beta(t) .$$

We have, of course, $\mu(\beta) = \mu(\beta')$ whenever $\beta \sim \beta'$. We can easily see that $\mu(\beta)$ can be also equivalently defined by means of the following property:

I. For each $\varepsilon > 0$ there exists a sequence of positive real numbers $\{t_j\}_{j=0}^{+\infty}$ with $\lim_{j \to +\infty} t_j = +\infty$ such that

1) the system of inequalities

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$$1 \le x \le (\mu(\beta)+\varepsilon)t \;;\; |x\beta - y| < \frac{1}{t}$$

has for each real $t > t_0$ at least one solution in integral numbers x, y,

2) none of the systems of inequalities

$$1 \le x \le (\mu(\beta)-\varepsilon)t_j \;;\; |x\beta - y| < \frac{1}{t_j} \;,\; j = 1, 2, 3, \ldots$$

has any solutions in integral numbers x, y.

Equally easily we can see that the value $\mu(\beta)$ can be fully characterized by the following, seemingly different property:

II. For each $\varepsilon > 0$ there exists a sequence of natural numbers $\{n_j\}_{j=0}^{+\infty}$ with $\lim_{j\to+\infty} n_j = +\infty$ such that

1) the system of inequalities

$$1 \le x \le (\mu(\beta)+\varepsilon)n \;;\; |x\beta - y| < \frac{1}{n}$$

has for each natural number $n > n_0$ at least one solution in integral numbers x, y,

2) none of the systems of inequalities

$$1 \le x \le (\mu(\beta)-\varepsilon)n_j \;;\; |x\beta - y| < \frac{1}{n_j} \;,\; j = 1, 2, 3, \ldots$$

has any solutions in integral numbers x, y.

Possible values of $\mu(\beta)$ were studied already in a series of papers. It is useful to express the results in terms of $R_\beta = \dfrac{\mu(\beta)}{1-\mu(\beta)}$ ($R_\beta = +\infty$ when $\mu(\beta) = 1$). Here (see e.g. [2], [5])

$$R_\beta = \limsup_{k\to+\infty} (b_k; b_{k-1}, \ldots, b_1)\cdot(b_{k+1}; b_{k+2}, \ldots).$$

G. Szekeres [6] proved that $\mu(\beta) \ge \frac{1}{2} + \frac{1}{2\sqrt{5}}$ for all irrationals β (in his work another formulation was used).

Let $c_j = 1$ for all $j \geq 1$. S. Morimoto [5] has shown that

1) for $\beta \sim \alpha_0 = (\overline{1})$ or $\beta \sim \alpha_m = (\overline{2; c_1, c_2, \ldots, c_{2m-1}})$ for some $m \geq 1$ we have $R_\beta < 2 + \sqrt{5}$,

2) $2 + \sqrt{5}$ is the smallest accumulation point of the set $\mathfrak{M} = \bigcup_\beta \{R_\beta\}$,

3) there are only countably many β with $R_\beta < 2 + \sqrt{5}$.

J. Lesca [4] has shown that the condition $\beta \sim \alpha_m$ for some $m \geq 0$ is also necessary for the inequality $R_\beta < 2 + \sqrt{5}$. The authors [2] also proved some theorems which characterize the set $\mathfrak{M}$. We could, of course, reformulate each of these results in terms of diophantine approximations according to I. and II. The object of this paper is to prove two theorems concerning the solvability of the system

$$1 \leq x \leq \mu(\beta) n \,;\; |x\beta - y| < \frac{1}{n}$$

in certain particular cases.

From what was said above it follows that the system of inequalities ($\mu(\beta) = \frac{1+\sqrt{5}}{4} \Longleftrightarrow R_\beta = 2 + \sqrt{5}$)

$$(1) \qquad 1 \leq x \leq \frac{1+\sqrt{5}}{4} n \,;\; |x\beta - y| < \frac{1}{n}$$

has a solution in integral numbers x, y for all sufficiently large natural numbers n, whenever there exists an integral number $m \geq 0$ such that $\beta \sim \alpha_m$. A natural question arises, namely, if there is also some $\beta \not\sim \alpha_m$ $(m = 0, 1, 2, \ldots)$ with this property.

The answer is as follows.

Theorem 1. There exist uncountably many numbers β (thus, also transcendental) such that for every natural number $\kappa > 1$ the system of inequalities (1) has at least one solution in integral numbers x, y.

Proof. 1. We shall seek β in the form

$$(2)\quad \begin{cases} \beta = (b_0; b_1, b_2, \dots) = (2; c_1, c_2, \dots, c_{2n_1-1}, \\ \qquad 2, c_1, c_2, \dots, c_{2n_2-1}, 2, c_1, c_2, \dots, c_{2n_3-1}, 2, c_1, \dots), \end{cases}$$

where

(3) $1 = n_1 < n_2 < n_3 < \dots$ and $c_j = 1$ for $j \geq 1$.

Hence, we have $2 = b_0 = b_{N_1} = b_{N_2} = \dots$, where $N_k = 2\sum_{j=1}^{k} n_j$ $(k \geq 1)$, and $b_j = 1$ otherwise. For each integral number $n \geq 0$ we denote by $\frac{p_n}{q_n} = (b_0; b_1, b_2, \dots, b_n)$, where $(p_n, q_n) = 1$ and $q_n \geq 1$ the n-th convergent of β. For $n \geq 1$ we put $x_n = (b_n; b_{n+1}, \dots)$. It is well known that then $\beta = \dfrac{x_{n+1} p_n + p_{n-1}}{x_{n+1} q_n + q_{n-1}}$ for $n \geq 1$ and hence,

$|q_n \beta - p_n| = \dfrac{1}{x_{n+1} q_n + q_{n-1}}$ for $n \geq 1$. It follows that for $n \geq 1$ the numbers $x = q_n$, $y = p_n$ are a solution of the system (1) for

(4) $$(\sqrt{5} - 1) q_n \leq \kappa < x_{n+1} q_n + q_{n-1} .$$

2. Now, we show that $(\sqrt{5}-1) q_{n+1} < x_{n+1} q_n + q_{n-1}$

for $n \geq 1$ and $n \neq N_k - 1$ $(k \geq 1)$. We have, namely, $q_{n+1} = q_n + q_{n-1}$ and $b_{n+1} = 1$ for such n, and hence

$$(\sqrt{5}-1)q_{n+1} - x_{n+1}q_n - q_{n-1} =$$
$$= (\sqrt{5}-1)q_{n+1} - x_{n+1}q_n - q_{n+1} + q_n =$$
$$= (\sqrt{5}-2)q_{n+1} - (x_{n+1}-1)q_n =$$
$$= q_n(x_{n+1}-1)(\sqrt{5}-2)\left(\frac{1}{x_{n+1}-1}\,\frac{q_{n+1}}{q_n} - \frac{1}{\sqrt{5}-2}\right) =$$
$$= q_n(x_{n+1}-1)(\sqrt{5}-2)\left(x_{n+2}\frac{q_{n+1}}{q_n} - \sqrt{5}-2\right).$$

It suffices to show that

$$x_{n+2}\frac{q_{n+1}}{q_n} = (b_{n+1}; b_n, \ldots, b_1)(b_{n+2}; b_{n+3}, \ldots) < 2 + \sqrt{5}.$$

This is clear for $b_{n+2} = 1$. Thus, let $b_{n+2} = 2$, i.e. $n = N_\ell - 2$ for some $\ell \geq 2$. We have then

$$x_{n+2}\frac{q_{n+1}}{q_n} = (b_{N_\ell-1}; b_{N_\ell-2}, \ldots, b_1)(b_{N_\ell}; b_{N_\ell+1}, \ldots) <$$

$$< (b_{N_\ell-1}; b_{N_\ell-2}, \ldots, b_{N_{\ell-1}})(b_{N_\ell}; b_{N_\ell+1}, \ldots, b_{N_{\ell+1}-2}, b_{N_{\ell+1}-1}) \leq$$
$$\leq (c_1; c_2, \ldots, c_{2m_\ell-1}, 2)(2; c_1, c_2, \ldots, c_{2m_\ell-1}, 1, 1) =$$
$$= \frac{a_{2m_\ell+1}}{a_{2m_\ell}}\,\frac{a_{2m_\ell+3}}{a_{2m_\ell+1}} = \frac{a_{2m_\ell+3}}{a_{2m_\ell}} = 2\,\frac{a_{2m_\ell+1}}{a_{2m_\ell}} + 1 <$$
$$< 2\,\frac{1+\sqrt{5}}{2} + 1 = 2 + \sqrt{5},$$

where $a_0 = a_1 = 1$, $a_{j+2} = a_{j+1} + a_j$ for $j \geq 0$ (Fibonacci numbers).

3. In order to finish the proof of Theorem 1, it remains for us to show that there are (uncountably many) sequences $1 = m_1 < m_2 < m_3 < \ldots$ such that between the numbers $(\sqrt{5}-1)q_{N_k}$, $x_{N_k}q_{N_k-1} + q_{N_k-2}$ $(k \geq 1)$

does not lie any natural number. We can easily see that the initial choice $n_1 = 1$ was good. For, we have $N_1 = 2$ and thus $q_2 = 3$, $q_1 = q_0 = 1$ and $x_2 > 2$, from which follows $(\sqrt{5}-1)q_2 < 4$ and $x_2 q_1 + q_0 > 3$.

Let us suppose now that we have already suitably chosen numbers $n_1, n_2, \dots, n_s$ $(s \geq 1)$ and we show that also n_{s+1} can be suitably chosen (in infinitely many ways). Firstly, we shall consider the difference $(\sqrt{5}-1)q_{N_{s+1}} - x_{N_{s+1}+1} - q_{N_{s+1}-2}$. Using again the notation a_j for Fibonacci numbers, we can easily verify that

$$q_{N_{s+1}} = a_{2n_{s+1}+1}\, q_{N_s} + a_{2n_{s+1}}\, q_{N_s-1} \quad ,$$

$$q_{N_{s+1}-1} = a_{2n_{s+1}-1}\, q_{N_s} + a_{2n_{s+1}-2}\, q_{N_s-1} \quad ,$$

$$(5) \qquad q_{N_{s+1}-2} = q_{N_{s+1}} - 2q_{N_{s+1}-1} = a_{2n_{s+1}-2}\, q_{N_s} + a_{2n_{s+1}-3}\, q_{N_s-1} \; ,$$

$$a_{2j}\sqrt{5} = \alpha_0^{2j-1} + \alpha_0^{-2j+1} \quad (j \geq 0) \quad ,$$

$$a_{2j+1}\sqrt{5} = \alpha_0^{2j+2} - \alpha_0^{-2j-2} \quad (j \geq 0) \quad ,$$

where $\alpha_0 = (\overline{1}) = \dfrac{1+\sqrt{5}}{2}$.

By means of (5), we can write:

$$(\sqrt{5}-1)q_{N_{s+1}} - x_{N_{s+1}}\, q_{N_{s+1}-1} - q_{N_{s+1}-2} = (\sqrt{5}-2)q_{N_{s+1}} -$$

$$- (x_{N_{s+1}} - 2)q_{N_{s+1}-1} = \alpha_0^{-3}\frac{1}{\sqrt{5}}\,[(\alpha_0^{2n_{s+1}+2} - \alpha_0^{-2n_{s+1}-2})q_{N_s} +$$

$$+\left(\alpha_0^{2m_{s+1}+1}+\alpha_0^{-2m_{s+1}-1}\right)q_{N_s-1}\Big] -$$

$$-\alpha_0^{-1}\frac{1}{\sqrt5}\Big[\left(\alpha_0^{2m_{s+1}}-\alpha_0^{-2m_{s+1}}\right)q_{N_s}+\left(\alpha_0^{2m_{s+1}-1}+\alpha_0^{-2m_{s+1}+1}\right)q_{N_s-1}\Big] -$$

$$-\left(x_{N_{s+1}}-\alpha_0^{-1}-2\right)q_{N_{s+1}-1} =$$

$$=\frac{1}{\sqrt5}\left(\alpha_0^{-1}-\alpha_0^{-5}\right)\alpha_0^{-2m_{s+1}}\left(q_{N_s}-\alpha_0 q_{N_s-1}\right)-\left(x_{N_{s+1}}-\alpha_0-1\right)q_{N_{s+1}-1} <$$

$$< \frac{1}{\sqrt5}\left(\alpha_0^{-1}-\alpha_0^{-5}\right)\alpha_0^{-2m_{s+1}}\left(q_{N_s}-\alpha_0 q_{N_s-1}\right),$$

since $x_{N_{s+1}} > \alpha_0 + 1$.

Hence, we have the following result:

$$(6)\qquad (\sqrt5-1)q_{N_{s+1}}-x_{N_{s+1}-1}\; q_{N_{s+1}-2} < C_s\,\alpha_0^{-2m_{s+1}} ,$$

where the positive constant

$$C_s = \frac{1}{\sqrt5}\left(\alpha_0^{-1}-\alpha_0^{-5}\right)\left(q_{N_s}-\alpha_0\, q_{N_s-1}\right)$$

depends only on $m_1, m_2, \dots, m_s$.

4. Now, we shall consider the expression

$$(\sqrt5-1)q_{N_{s+1}}-q_{N_{s+1}}-q_{N_{s+1}-2} .$$

Again, by means of the formula (5), we can write:

$$(\sqrt5-1)q_{N_{s+1}}-q_{N_{s+1}}-q_{N_{s+1}-2}=(\sqrt5-2)q_{N_{s+1}}-q_{N_{s+1}-2} =$$

$$=\alpha_0^{-3}\frac{1}{\sqrt5}\Big[\left(\alpha_0^{2m_{s+1}+2}-\alpha_0^{-2m_{s+1}-2}\right)q_{N_s}+\left(\alpha_0^{2m_{s+1}+1}+\alpha_0^{-2m_{s+1}-1}\right)q_{N_s-1}\Big]-$$

$$-\frac{1}{\sqrt5}\Big[\left(\alpha_0^{2m_{s+1}-1}+\alpha_0^{-2m_{s+1}+1}\right)q_{N_s}+\left(\alpha_0^{2m_{s+1}-2}-\alpha_0^{-2m_{s+1}+2}\right)q_{N_s-1}\Big]=$$

$$=-\frac{1}{\sqrt5}\left(\alpha_0+\alpha_0^{-5}\right)\alpha_0^{-2m_{s+1}}\left(q_{N_s}-\alpha_0 q_{N_s-1}\right)=-C_s^*\,\alpha_0^{-2m_{s+1}} ,$$

where the positive constant $C_s^* = \frac{1}{\sqrt5}\left(\alpha_0+\alpha_0^{-5}\right)\left(q_{N_s}-\alpha_0\, q_{N_s-1}\right)$

depends only on the numbers $n_1, n_2, \dots, n_s$.

5. As to n_{s+1} we may choose any natural number satisfying the conditions $n_{s+1} > n_s$ and $(C_s + C_s^*)\alpha_0^{-2n_{s+1}} < 1$ or, since $C_s + C_s^* = q_{N_s} - \alpha_0 q_{N_s - 1}$, the inequalities

$$(7) \qquad n_{s+1} > n_s, \; n_{s+1} > \frac{\log (q_{N_s} - \alpha_0 q_{N_s-1})}{2 \log \alpha_0} .$$

Any such chosen n_{s+1} will be suitable in the sense that no natural number k will satisfy

$$(\sqrt{5} - 1) q_{N_{s+1}} < k < x_{N_{s+1}} q_{N_{s+1}-1} + q_{N_{s+1}-2} .$$

This follows from (6), from an equality derived in Part 4 of this proof and from the condition (7). Since this construction can be indefinitely continued, Theorem 1 is proved.

<u>Remark</u>. It is almost trivial that not all irrational numbers β with $\mu(\beta) = \frac{\sqrt{5}+1}{4}$ have the property pointed out in Theorem 1. This follows immediately from the relation

$$(\sqrt{5} - 1) q_{n+1} - x_{n+1} q_n - q_{n-1} = (\sqrt{5} - 2) q_{n+1} - (x_{n+1} - b_{n+1}) q_n =$$
$$= (\sqrt{5} - 2)(x_{n+1} - b_{n+1}) q_n \left(\frac{q_{n+1}}{q_n} x_{n+2} - \sqrt{5} - 2 \right) .$$

We can take β of the form (2), $\lim_{j \to +\infty} n_j = +\infty$, but not monotonically. By suitable "decreases" in the sequence $n_1, n_2, n_3, \dots$ we can arrange that between

$(\sqrt{5}-1)\,q_{n+1}$ and $x_{n+1}\,q_n + q_{n-1}$ there will be for infinitely many n a natural number.

Remark. From a related result of B. Diviš ([1], Theorem 2) it follows that there does not exist any irrational number β with $R_\beta = 2+\sqrt{5}$ such that the system of inequalities

$$1 \le x \le \frac{1+\sqrt{5}}{4}\,t\ ;\quad |x\beta - y| < \frac{1}{t}$$

has for each real $t \ge \frac{4}{1+\sqrt{5}} = \sqrt{5}-1$ a solution in integral numbers x, y.

Theorem 2. If θ is a quadratic irrationality then for all sufficiently large natural numbers κ the system of inequalities

$$(8)\qquad 1 \le x \le \mu(\theta)\kappa\ ;\quad |x\theta - y| < \frac{1}{\kappa}$$

has at least one solution in integral numbers x, y.

Proof. 1. We can write $\theta = (d_0\,;\,d_1, d_2, \dots) = (d_0\,;\,d_1,\dots,d_m,\overline{e_1,e_2,\dots,e_p})$, where $m \ge 1$ and $p \ge 2$ is an even number. We introduce the following notation:

$$\lambda_1 = (\overline{e_1\,;\,e_2,\dots,e_p}),\quad \lambda_2 = (\overline{e_2\,;\,e_3,\dots,e_p,e_1}),\ \dots,$$

$$\lambda_p = (\overline{e_p\,;\,e_1,e_2,\dots,e_{p-1}}),\quad \varkappa_1 = (\overline{e_1\,;\,e_p,e_{p-1},\dots,e_2})\ ,$$

$$\varkappa_2 = (\overline{e_2\,;\,e_1,e_p,e_{p-1},\dots,e_3}),\ \dots,\ \varkappa_p = (\overline{e_p\,;\,e_{p-1},\dots,e_1})\ .$$

Without loss of generality we may suppose that $R_\theta = \lambda_1\varkappa_p$ and thus $\lambda_{j+1}\varkappa_j \le \lambda_1\varkappa_p$ for $j = 1, 2, \dots, p-1$. Sometimes we shall use the symbol λ_n, where $n > p$.

This will mean $\lambda_n = \lambda_i$, where $n \equiv i \pmod{p}$ and $1 \leq i \leq p$. Analogously are the symbols $\varkappa_n$, e_n with $n > p$ to be understood. For $n \geq 1$ we denote by $\frac{p_n}{q_n} = (d_0; d_1, d_2, \ldots, d_n)$, where $(p_n, q_n) = 1$ and $q_n \geq 1$, the n-th convergent of the number θ. Then we have for $n > m$

$$|q_n \theta - p_n| = \frac{1}{q_n \lambda_{n+1} + q_{n-1}} .$$

From this equality it follows that for those natural numbers κ for which

$$q_n \frac{1}{\mu(\theta)} \leq \kappa < q_n \lambda_{n+1} + q_{n-1} ,$$

the pair $(x, y) = (q_n, p_n)$ is a solution of the system (8). It suffices to show that for sufficiently large natural numbers N there exists no natural number ℓ which would satisfy the relation

$$(9) \qquad q_N \lambda_{N+1} + q_{N-1} \leq \ell < q_{N+1} \frac{1}{\mu(\theta)} .$$

2. For $j \geq 1$ we have

$$q_{m+j} = e_j q_{m+j-1} + q_{m+j-2} .$$

From the periodicity of the sequence $\{e_j\}_{j=1}^{+\infty}$ ($e_j = e_{j+p}$, p even) it follows that there exist nonzero constants Λ (see equation (11)), A_k, B_k ($0 \leq k \leq p-1$) such that $q_{m+np+k} = A_k \Lambda^n + B_k \Lambda^{-n}$ for all $n \geq 0$ and $0 \leq k \leq p-1$. Moreover, we have

$$\Lambda = \lambda_1 \lambda_2 \dots \lambda_p = \varkappa_1 \varkappa_2 \dots \varkappa_p > 1 ,$$

$$A_k = \varkappa_1 \varkappa_2 \dots \varkappa_k A_0 , \quad k = 1, 2, \dots, p-1 ,$$

$$B_k = (-1)^k \frac{B_0}{\lambda_2 \lambda_3 \dots \lambda_{k+1}} , \quad k = 1, 2, \dots, p-1 .$$

The proof of these facts is a purely technical matter and may be left to the reader.

Sometimes we shall use the symbol A_j also when $j > p-1$. This will mean $A_j = A_k \Lambda^{n_0}$ where $j = n_0 p + k, \ 0 \leq k \leq p-1$. In a similar sense are the symbols B_j for $j > p-1$ to be understood.

3. Now consider $n \geq 2$ and $k \geq 0$. Since Λ is a quadratic irrationality and $\lambda_1, \varkappa_p, \Lambda$ are elements of the same quadratic number field, there exist two rational constants C_1, C_2 such that $\frac{1}{\lambda_1 \varkappa_p} = \frac{C_1}{\Lambda} + \frac{C_2}{\Lambda^2}$. Then we have

$$q_{m+np+k+2} \frac{1}{\mu(\theta)} = q_{m+np+k+2} \left(1 + \frac{1}{\lambda_1 \varkappa_p}\right) =$$

$$= (A_{k+2} \Lambda^n + B_{k+2} \Lambda^{-n}) \left(1 + \frac{C_1}{\Lambda} + \frac{C_2}{\Lambda^2}\right) =$$

$$= q_{m+np+k+2} + C_1 q_{m+(n-1)p+k+2} + C_2 q_{m+(n-2)p+k+2} -$$

$$- B_{k+2} \Lambda^{-n-2} (\Lambda^2 - 1)(C_2 \Lambda^2 + C_1 \Lambda + C_2) .$$

Further, let us consider the expression

$$q_{m+np+k+1} \lambda_{k+2} + q_{m+np+k} = q_{m+np+k+2} +$$

$$+\frac{1}{\lambda_{k+3}}\, q_{m+mp+k+1} = q_{m+mp+k+2} +$$

$$+\frac{1}{\lambda_{k+3}}\,(A_{k+1}\Lambda^{n} + B_{k+1}\Lambda^{-m}) = q_{m+mp+k+2} +$$

$$+\frac{1}{\lambda_{k+3}}\,(\frac{A_{k+2}}{\varkappa_{k+2}}\Lambda^{m} - B_{k+2}\lambda_{k+3}\Lambda^{-m}) .$$

If we have $\lambda_{k+3}\varkappa_{k+2} < \lambda_1 \varkappa_p$, then for sufficiently large m we have

$$q_{m+mp+k+2}(1+\frac{1}{\lambda_1\varkappa_p}) < q_{m+mp+k+1}\lambda_{k+2} + q_{m+mp+k}$$

and the inequality (9), obviously, has no solution for $N = m + mp + k + 1$ with sufficiently large m . Therefore, it suffices to consider the case, when

$$\lambda_{k+3}\varkappa_{k+2} = \lambda_1 \varkappa_p .$$

Then we have

$$q_{m+mp+k+1}\lambda_{k+2} + q_{m+mp+k} = q_{m+mp+k+2} +$$

$$+\frac{1}{\lambda_1\varkappa_p} A_{k+2}\Lambda^{m} - B_{k+2}\Lambda^{-m} = q_{m+mp+k+2} +$$

$$+ A_{k+2}\Lambda^{m}(\frac{C_1}{\Lambda} + \frac{C_2}{\Lambda^2}) - B_{k+2}\Lambda^{-m} =$$

$$= q_{m+mp+k+2} + C_1 q_{m+(m-1)p+k+2} + C_2 q_{m+(m-2)p+k+2} -$$

$$- B_{k+2}\Lambda^{-m}(C_2\Lambda^2 + C_1\Lambda + 1) .$$

4. Since $\Lambda > 1$, i.e. $\Lambda^{-m} \to 0$, it will be sufficient to show that the expressions $C_2\Lambda^2 + C_1\Lambda + C_2$ and $C_2\Lambda^2 + C_1\Lambda + 1$ have the same sign. We have, obviously,

$$\Lambda^2(C_2\Lambda^2+C_1\Lambda+1) > \Lambda^2(C_2\Lambda^2+C_1\Lambda-\frac{1}{\lambda_1\varkappa_p}) =$$
$$= C_2\Lambda^4+C_1\Lambda^3-C_1\Lambda-C_2 = (\Lambda^2-1)(C_2\Lambda^2+C_1\Lambda+C_2) =$$
$$= \Lambda^2(\Lambda^2-1)(C_2+\frac{1}{\lambda_1\varkappa_p}) .$$

Thus, it suffices only to show that

$$C_2 > -\frac{1}{\lambda_1\varkappa_p} . \tag{10}$$

For this purpose we shall need the following result, the proof of which is a purely technical matter and can be left again to the reader. For each even number p there exist four polynomials P, Q, Q_2, Q_3 with nonnegative integral coefficients in the variables $a_1, a_2, \ldots, a_p$ such that

$$\begin{gathered} \Lambda^2-(P+Q)\Lambda+1 = 0 , \\ \lambda_1 = \frac{\Lambda-Q}{Q_2} , \quad \varkappa_p = \frac{\Lambda-Q}{Q_3} , \\ C_2 = -Q_2Q_3\frac{Q^2-1}{(PQ-1)^2} , \end{gathered} \tag{11}$$

$$\Lambda > P > Q \geq 1 , \quad Q_2 \geq 1 , \quad Q_3 \geq 1 \tag{12}$$

for any system of natural numbers $a_1, a_2, \ldots, a_p$.
Then, the inequality (10) can be written as

$$-Q_2Q_3\frac{Q^2-1}{(PQ-1)^2} > -\frac{1}{\frac{(\Lambda-Q)^2}{Q_2Q_3}} ,$$

or $(Q^2-1)(\Lambda-Q)^2 < (PQ-1)^2$.

The proof of the last inequality by means of the relations (11), (12) does not represent any difficulty and can be left to the reader.

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