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SIZES OF SETS AND SOME FIXED POINT THEOREMS

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1. If x and y are distinct elements of a metric space (X, d) the distance $d(x, y)$ is a measure of the "size" of the set $\{x, y\}$ and a contraction mapping can be viewed as one which reduces, in a uniform way, the size of all two-element subsets. More generally, the diameter $d(A) = \sup \{d(x,y) : x, y \in A\}$ is a measure of the size of an arbitrary bounded subset A of X. A further notion of the size of an arbitrary bounded subset A of X was introduced by Kuratowski [11] as the infimum of those positive numbers ε such that A can be covered by a finite number of subsets of X of diameter less than ε. The closely related notion of the infimum of those ε such that A can be covered by a finite ε-net has been considered by Sadovskii [12] for Banach spaces. Sadovskii and earlier Darbo [5] - who used Kuratowski's definition - proved fixed point theorems for mappings which reduce the size of bounded subsets of the spaces considered by them.

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The purpose of this note is to present the fixed point theorem of Browder-Göhde-Kirk [3, 6, 10] in a similar light and to give a proof of the Darbo-Sadovskii theorem using Bourbaki's fixed point theorem instead of Zorn's Lemma.

2. A type of uniform structuree

Let X be a topological space. Let $\mathcal{B}$ be a family of subsets of X such that (i) $X \in \mathcal{B}$, (ii) $B \in \mathcal{B}$ implies $\bar{B} \in \mathcal{B}$, (iii) the intersection of members of any subfamily of $\mathcal{B}$ belongs to $\mathcal{B}$.

The family of all closed subsets of a topological space and the family of all closed convex subsets of a linear topological space are simple examples of such families.

Definition 2.1. A $\mathcal{B}$-uniform structure on X is a family $\mathcal{U}$ of subsets of $X \times X$ such that

(i) $\Delta \subseteq U$ for every $U \in \mathcal{U}$; $(\Delta = \{(x,x) : x \in X\})$ and $X \times X \in \mathcal{U}$.

(ii) Every $U \in \mathcal{U}$ is symmetric.

(iii) For every $x \in X$ and $U \in \mathcal{U}$, the set $U_x = (\{x\} \times X) \cap U$ is closed and belongs to $\mathcal{B}$.

(iv) For every $x \in X$, $\bigcap \{U_x : U \in \mathcal{U}\} = \{x\}$.

We write $\mathcal{U}_x$ to denote the collection $\{U_x : U \in \mathcal{U}\}$.

Remarks:

(a) We do not require all the properties of Hausdorff uniformity in the usual sense.

(b) The only connection between the topology and the uniform structure is that specified in (iii). This allows, for example, the possibility of X being a normed linear space with the weak topology, $\mathcal{B}$ the family of all closed convex sets and the $\mathcal{B}$-uniform structure being such that $\mathcal{U}_x$ is the family of all closed balls with positive radius centered at x.

(c) "Symmetric" in (ii) means that for all $x, y \in U$, if $y \in U_x$, then $x \in U_y$.

(d) Conditions (iii) and (iv) together imply some further conditions on the family $\mathcal{B}$, for example: $\{x\} \in \mathcal{B}$ for every $x \in X$.

3. Notions of the "size" of sets.

Let X be a topological space with a $\mathcal{B}$-uniform structure $\mathcal{U}$. For a subset A of X we consider the following families of subsets of $\mathcal{U}$.

Definition 3.1.

(i) $D(A) = \{U \in \mathcal{U}$: for every $x \in A, A \subseteq U_x\}$;

(ii) $R(A) = \{U \in \mathcal{U}$: there exists $x \in A$ with $A \subseteq U_x\}$;

(iii) $Q_1(A) = \{U \in \mathcal{U}$: there exists a finite subset F of X with $A \subseteq \bigcup\{U_x : x \in F\}\}$;

(iv) $Q_2(A) = \{U \in \mathcal{U}$: there exists a finite subset F of A with $A \subseteq \bigcup\{U_x : x \in F\}\}$.

These sets are non-empty since $X \times X$ belongs to each one of them.

The set D is called the $\mathcal{U}$-diameter of A; R is called the $\mathcal{U}$-A-radius of A and depends on the "shape" of A since points x in A are needed for the "centers" of the measuring sets $\mathcal{U}_x$; Q_i are measures of the "total boundedness" (or precompactness) of A relative to $\mathcal{U}$. It should be noted that a measure of non-compactness defined by means of " ε -nets" was introduced in [7] and [8], and also, independently, in [12].

Observe that when X is a normed linear space and $\mathcal{U}_x$ is the family of all closed balls of positive radius centered at x , we can identify each $U \in \mathcal{U}$ with a ball of positive radius centered at the origin. Then, for bounded subsets A of X ,

$d(A) = \inf\{\varepsilon : B(0,\varepsilon) \in D(A)\}$ is the diameter of A ,

$\kappa(A) = \inf\{\varepsilon : B(0,\varepsilon) \in R(A)\}$ is the "radius" of A ,

and

$q_i(A) = \inf\{\varepsilon : B(0,\varepsilon) \in Q_i(A)\}$ $(i = 1, 2)$ is such that for all $\varepsilon > q_i(A)$ there exists a finite ε -net for A (with elements in X and in A respectively).

We further remark that in this context there is a semi-linear structure available for bounded subsets of X .

$$A_1 + A_2 = \{x_1 + x_2 : x_i \in A_i\}, \ \lambda A = \{\lambda x : x \in A\} (\lambda \geq 0)$$

and that with this structure d, κ and q_i are "semi-norms" in the sense that

$$d(A_1 + A_2) \leq d(A_1) + d(A_2), \ d(\lambda A) = \lambda d(A) \ (\lambda \geq 0)$$

and similarly, for κ and q_i . Moreover, d and q_1 are monotonic, i.e., if $A_1 \subseteq A_2$ then $d(A_1) \leq d(A_2)$. Also the q_i satisfy the inequalities

$$q_1(A) \leq q_2(A) \leq 2q_1(A) \quad \text{for all } A \subseteq X .$$

Definition 3.2. The set A is said to be $\mathcal{U}$-totally bounded if $Q_1(A) = \mathcal{U}$.

Definition 3.3. The set A has $\mathcal{U}$-normal structure if $R(B) \supset D(B)$ for every B such that $B \in \mathcal{B}$, $B \subseteq A$ and B is not a singleton.

Remarks:

(i) We use $\subset$ (or $\supset$) to mean strict inclusion throughout.

(ii) The set A is a singleton if and only if $D(A) = \mathcal{U}$.

(iii) The definition of normal structure comes directly from the definition of Brodskii and Milman [2] for a bounded convex subset of a normed linear space, namely, A has normal structure if and only if $\kappa(C) < d(C)$ for every non-trivial convex subset C of A .

Next we are concerned with fixed point theorems for mappings which satisfy inequalities with respect to the above measures of size. In what follows we shall usually suppress explicit reference to $\mathcal{U}$ and speak of normal structure, diameter, etc.

4. The Browder-Göhde-Kirk fixed point theorem.

Let X be a topological space with a $\mathcal{B}$-uniform structure. Let f be a function which maps X into itself.

Definition 4.1. (i) f is non-expansive if

$D(f(A)) \geq D(A)$ for every two-element set $A = \{x, y\} \subseteq X$.

(ii) f is normalizing if $R(ch\, f(B)) \supset D(B)$ for every non-trivial subset B of $\mathcal{B}$.

Here $ch\, A$ (with respect to $\mathcal{B}$) denotes the smallest closed member B of $\mathcal{B}$ which contains A (and is the intersection of all such B). We observe also that $R(ch\, A) \supseteq R(A)$ and is, in general, larger since there may be suitable "centers" in $ch\, A$ which are not in A .

Theorem 1. Let X be a compact topological space with a $\mathcal{B}$-uniform structure. Let f be a mapping of X into itself, which is both normalizing and non-expansive. Then f has a fixed point.

Proof: (similar to that in [10]) Consider the set $\mathcal{P}$ of all non-empty closed subsets A of X which are in $\mathcal{B}$ and which are invariant under f . The set $\mathcal{P}$ is non-empty since X belongs to it. Also it is inductively ordered by inclusion since any chain in $\mathcal{P}$ has the finite intersection property and hence has a lower bound (the intersection of all members of the chain) which is non-empty since X is compact and belongs to $\mathcal{B}$ since $\mathcal{B}$ is stable for such intersections. Hence, by Zorn's Lemma, $\mathcal{P}$ has minimal elements. If A_0 is such a minimal element it is a fixed point of the mapping φ defined by $\varphi(A) = ch\,(f(A))$. Suppose, if possible, that A_0 is not a singleton, i.e. $D(A_0) \neq \mathcal{U}$. Then, since f is normalizing, $R(g(A_0)) = R(A_0) \supset D(A_0)$. Let $U' \in \mathcal{U}$ be chosen so that $U' \notin D(A_0)$ but $U' \in R(A_0)$ and let $A_1 = \{x \in A_0 : A_0 \subseteq U'_x\}$. Then (i) A_1 is non-empty since $U' \in R(A_0)$ and so there exists an x such that $A_0 \subseteq U'_x$; (ii) $A_1 \subseteq A_0$ by definition; (iii) $A_1 \neq A_0$ since $U' \notin D(A_0)$ and so not every x in A_0 belongs to A_1; (iv) $A_1 = A_0 \cap \{U'_y ; y \in A_0\}$ for, if $x \in A_1$, then $y \in U'_x$ for all $y \in A_0$ and hence by the symmetry of U', $x \in U'_y$ for all y in A_0, i.e., $x \in A_0 \cap \{U'_y : y \in A_0\}$. Conversely, if x is in the intersection then $x \in U'_y$ for all $y \in A_0$ and so $y \in U'_x$ for all $y \in A_0$, i.e., $A_0 \subseteq U'_x$.

Now since A_0 and U'_y are all members of $\mathcal{B}$, (iv) implies that $A_1 \in \mathcal{B}$. Moreover, since $A_0 \in \mathcal{P}$ and since U'_y is closed for each y, A_1 is closed. We prove, finally, that A_1 is invariant under f. Let $x \in A_1$, then $A_0 \subseteq U'_x$, i.e., $y \in U'_x$ for all $y \in A_0$ and hence also, $x \in U'_y$ so that $U' \in D\{x, y\}$ for all $x \in A_1$, $y \in A_0$. Now f is non-expansive so that $U' \in D\{f(x), f(y)\}$ for all $x \in A_1$, $y \in A_0$, i.e., $U'_{f(x)} \supseteq f(A_0)$. But then, since $U'_{f(x)}$ is a closed member of $\mathcal{B}$, $U'_{f(x)} \supseteq ch\, f(A_0) = g(A_0) = A_0$. Thus $f(x) \in A_1$ and A_1 is invariant under f and hence is in $\mathcal{P}$. But (ii) and (iii) contradict the minimality of A_0. Thus A_0 consists of a single point $\{a_0\}$. Hence $ch\{f(a_0)\} = \{a_0\}$. This implies that $f(a_0) = a_0$ because, for any $b_0 \in X$, $\{b_0\} \subseteq ch\{b_0\} = \{B \in \mathcal{B} : b_0 \in B$ and B is closed$\} \subseteq \bigcap\{U_{b_0} : U \in \mathcal{U}\} = \{b_0\}$.

The purpose of the next lemma and proposition is to establish a connection between normalizing mappings and sets with normal structure.

<u>Lemma 4.2.</u> If X is a topological space with $\mathcal{B}$-uniform structure and if A is a subset of X , then (i) $D(ch\ A) = D(A)$ and (ii) if $f: X \to X$ is non-expansive, then $D(f(A)) \supseteq D(A)$.

<u>Proof</u>: (i) Since $A \subseteq ch\ A$ it is clear that if $U \in D(ch\ A)$ then $U \in D(A)$. Suppose conversely that $U \in D(A)$. Let $x \in A$. Then $A \subseteq U_x$ and so, since $U_x \in \mathcal{B}$ and is closed, $ch\ A \subseteq U_x$. Thus, if $y \in ch\ A$, $y \in U_x$ and hence $x \in U_y$ for every $x \in A$, i.e., $A \subseteq U_y$ and so $ch\ A \subseteq U_y$ for every $y \in ch\ A$, i.e., $U \in D(ch\ A)$.
(ii) Let $U \in D(A)$, then $U \in D\{x, y\}$ for every pair $x, y \in A$. Since f is non-expansive, this means $U \in D\{f(x), f(y)\}$ for every pair $x, y \in A$, i.e., $U \in D\{u, v\}$ for every pair $u, v \in f(A)$, i.e., $u \in U_v$ for every $u, v \in f(A)$. Thus $f(A) \subseteq U_v$ for every $v \in f(A)$ and so $U \in D(f(A))$.

<u>Remark</u>: Part (ii) of the lemma means that the restriction to pairs in the definition of non-expansive mappings is unnecessary.

<u>Proposition 4.3</u>. If X has normal structure, then every non-expansive mapping of X into itself is normalizing.

Proof: For every non-trivial subset $\mathcal{B}$ of A for which $\mathcal{B} \in \mathfrak{B}$, it is true that $\mathcal{R}(ch\, f(\mathcal{B})) \supset \supset \mathcal{D}(ch\, f(\mathcal{B}))$, by the definition of normal structure, and by Lemma 4.2, it follows that $\mathcal{R}(ch\, f(\mathcal{B})) \supset \mathcal{D}(f(\mathcal{B})) \supseteq \mathcal{D}(\mathcal{B})$.

5. The Darbo-Sadovskii fixed point theorem.

As in Section 4, let X be a topological space with a $\mathfrak{B}$-uniform structure $\mathcal{U}$ and let f be a mapping of X into itself. In this section, however, we shall assume more: namely that $\mathcal{U}$ is a base for a uniformity on X and that the topology on X is the uniform topology generated by $\mathcal{U}$. The definition (3.2 above) of $\mathcal{U}$-totally bounded sets then coincides with the more usual definition (see for example [9], p.198). We shall also assume throughout this section that the mapping f is continuous. For the main theorem we make use of Tychonoff's fixed point theorem and, therefore, require a linear structure and a specific type of $\mathfrak{B}$-uniformity; for this reason this section is somewhat different in character from the preceding ones.

Definition 5.1. The mapping f is condensing on X if $\mathcal{Q}_1(f(A)) \supset \mathcal{Q}_1(A)$ for every A in X which is not totally bounded.

The following lemma is clear.

Lemma 5.2. If f is condensing on X and if $A \subseteq \subseteq X$ and $f(A) = A$ then A is totally bounded.

Lemma 5.3. If f is condensing on X then every orbit $O(x) = \{f^n(x) \mid n = 1, 2, 3, \ldots\}$ is totally bounded.

Proof: Since $f(O(x)) \subseteq O(x)$ we have $\alpha_1(f(O(x)) \supseteq \alpha_1(O(x))$. On the other hand, since $O(X) = f(O(x)) \cup \{x\}$, if F is a finite $\mathcal{U}$ -net for $f(O(x))$, then $F \cup \{x\}$ is a finite $\mathcal{U}$ -net for $O(x)$. Hence $\alpha_1(O(x)) = \alpha_1(f(O(x)))$ and we must have $O(x)$ totally bounded.

Now suppose that X is complete with respect to the uniformity $\mathcal{U}$. It is well known ([9]p.198) that if A is totally bounded and complete then A is compact. It is, therefore, a corollary to Lemma 5.3, that if X is complete and if f is condensing on X then $K = \overline{O(x)}$ is a compact subset of X for all x in X . Moreover, since we are assuming that f is continuous, $f(K) \subseteq K$. Thus, there exist non-empty compact subsets of X which are invariant under f . The purpose of the rest of this section is to show that when $\mathcal{B}$ is the family of convex sets in a locally convex linear topological space X , then there are compact elements of $\mathcal{B}$ which are invariant under f . The argument is divided into two lemmas.

Lemma 5.4. Let $\mathcal{B}$ be the family of convex sets and $\mathcal{U}$ a uniformity for the topology of a locally convex linear topological space X generated by closed convex neighborhoods of 0 , then $\alpha_1(ch\ A) = \alpha_1(A)$ for each A in X .

This lemma is proved in the same way as Lemma 1 of [12].

<u>Lemma 5.5.</u> With the hypotheses of the preceding lemma, and if f is a condensing mapping on some element A of $\mathcal{B}$ which is complete then there is a compact element of $\mathcal{B}$ invariant under f.

<u>Proof</u>: Let K denote the closure of an orbit $\overline{O(x)}$. Let $\mathcal{P}$ be the set of all closed non-empty elements $C \in \mathcal{B}$ such that $f(C) \subseteq C$ and $C \cap K \neq \emptyset$ and consider $\mathcal{P}$ partially ordered by inclusion. Then P is non-empty since, $A \in \mathcal{P}$. Consider the mapping $f : \mathcal{P} \longrightarrow \mathcal{P}$ defined by $g(C) = ch(f(C))$. Then, since C is closed and convex (in $\mathcal{B}$) and $f(C) \subseteq C$, we have $g(C) \subseteq C$. Moreover every chain in $\mathcal{P}$ has a lower bound $L = \bigcap\{C : C$ in the chain$\}$. ($L \cap K$ is non-empty since, for each C in the chain, $C \cap K$ is a non-empty compact set and these sets have the finite intersection property, and, clearly, L has the other defining properties of $\mathcal{P}$.) By the Bourbaki fixed point theorem (see, for example, [1] p.41) the mapping g has a fixed point in $\mathcal{P}$; i.e., $g(C_0) = C_0$. Thus $ch\, f(C_0) = C_0$. Now

$$\alpha_1(ch\, f(C_0)) = \alpha_1(f(C_0)) \supseteq \alpha_1(C_0) = \alpha_1(ch\, f(C_0))$$

with the inclusion strict, by the condensing property of f, unless C_0 is totally bounded. Thus C_0 is totally bounded. Since C_0 is also a closed subset of the complete set A, C_0 is compact.

<u>Theorem 2</u> (Darbo-Sadovskii). Let A be a complete, convex, bounded subset of a locally convex linear

topological space E and let f be continuous and condensing on A. Then f has a fixed point in A.

<u>Proof</u>. By Lemma 5.5 there is a compact convex subset C_0 of A invariant under f. The result now follows from Tychonoff's fixed point theorem applied to f and the set C_0.

<u>Remark</u>: Lemma 5.5 can be obtained from Propositions 1 and 4 of [4], and Theorem 2 is contained in Theorem 3 of [4]. However the above proof does not use Zorn's Lemma.

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