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A REMARK ON SCHWARTZ SPACES CONSISTENT WITH A DUALITY

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In [3], a notion of $(\mathcal{S})$-neighbourhood in a topological linear space E was introduced. We show that a more simple condition is sufficient for a neighbourhood to be $(\mathcal{S})$-neighbourhood and use this result to obtain a simpler characterization of the finest Schwartz topology consistent with a duality.

Let $\mathcal{U}$ be a closed absolutely convex subset of E. Then $E_{\mathcal{U}}$ denotes the normed space obtained by taking $\mathcal{U}$ as closed unit ball in the vector space generated by $\mathcal{U}$ and passing to a factor space, if the topology is not separated. By $E(\mathcal{U}, \mathcal{V})$ we mean the continuous map $E_{\mathcal{U}} \longrightarrow E_{\mathcal{V}}$ induced from the identity transformation of E, if $\mathcal{U} \subset \mathcal{V}$. By a neighbourhood we always mean a closed absolutely convex neighbourhood of zero.

In this notation a neighbourhood $\mathcal{U}$ is called $(\mathcal{S})$-neighbourhood in E, if there exists a sequence $\{\mathcal{U}_n\}$ of neighbourhoods in E such that $\mathcal{U}_0 = \mathcal{U}$, $\mathcal{U}_{n+1} \subset \mathcal{U}_n$ and $E(\mathcal{U}_{n+1}, \mathcal{U}_n)$ is completely continuous map of $E_{\mathcal{U}_{n+1}}$ into $E_{\mathcal{U}_n}$ for all n. The fol-

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lowing proposition says that only the existence of one such neighbourhood $\mathcal{U}_1 \subset \mathcal{U}$ is sufficient.

1. Proposition. A neighbourhood $\mathcal{U}$ in the topological space E is (S)-neighbourhood if and only if there is a neighbourhood V in E, $V \subset \mathcal{U}$, such that the operator $E(V, \mathcal{U}) : E_V \longrightarrow E_{\mathcal{U}}$ is completely continuous.

Proof. The restricted condition is obviously necessary for $\mathcal{U}$ to be a (S)-neighbourhood. To prove the converse we use the following proposition [4, Proposition 3]:

An operator $T : E \longrightarrow F$ (E and F are normed spaces) is completely continuous and $|T| \leq \beta$ if and only if for every $\varepsilon > 0$ there exists a sequence $\{a_n\}$, $a_n \in E'$, $|a_n| \leq \beta + \varepsilon$, $|a_n| \longrightarrow 0$ such that $|T(x)| \leq \sup |a_n(x)|$ for every $x \in E$.

Now if we have a neighbourhood V such that $E(V, \mathcal{U}) : E_V \longrightarrow E_{\mathcal{U}}$ is completely continuous, then, using this proposition we obtain the existence of $a_n \in E'_V$, $|a_n| \leq 1$ and $\alpha_n > 0$, $\alpha_n \longrightarrow 0$ such that $p_{\mathcal{U}}(x) = |E(V, \mathcal{U})| \leq \sup \alpha_n |a_n(x)|$.
We prove first that there is a neighbourhood W in E such that $V \subset W \subset \mathcal{U}$ and the operators $E(V, W)$ and $E(W, \mathcal{U})$ are completely continuous. It is sufficient to put $W = \{x \in E \mid \sqrt{\alpha_n} \, |a_n(x)| \leq 1\}$ i.e. the polar set of the bounded set $\{b_n\} \subset E'_V$, where $b_n = \sqrt{\alpha_n} \, a_n$. W is obviously a neighbourhood in E and $p_W(x) = \sup |\sqrt{\alpha_n} \, a_n(x)|$. Using again the above

mentioned proposition, we obtain that $E(\mathcal{V},W): E_{\mathcal{V}} \to \to E_W$ is completely continuous. To see that also $E(W,\mathcal{U}): E_W \to E_{\mathcal{U}}$ is completely continuous, we observe that

$$p_{\mathcal{U}}(x) \leq \sup \alpha_n |a_n(x)| = \sup \sqrt{\alpha_n} |b_n(x)|$$

and $p_W(b_n) \leq 1$.

Now we put $\mathcal{U}_1 = W$. The operator $E(\mathcal{V}, \mathcal{U}_1)$ being completely continuous, we may, by the same reason, find a neighbourhood $\mathcal{U}_2$ in E, $\mathcal{V} \subset \mathcal{U}_2 \subset \mathcal{U}_1$ such that $E(\mathcal{V}, \mathcal{U}_2)$ and $E(\mathcal{U}_2, \mathcal{U}_1)$ are completely continuous. Proceeding by induction we obtain a sequence $\{\mathcal{U}_n\}$ of neighbourhoods in E, $\mathcal{V} \subset \mathcal{U}_{n+1} \subset \mathcal{U}_n \subset \mathcal{U}_0 = \mathcal{U}$, such that $E(\mathcal{U}_{n+1}, \mathcal{U}_n)$ and $E(\mathcal{V}, \mathcal{U}_{n+1})$ are completely continuous. This proves our proposition.

2. <u>Proposition</u>. Let E, F be paired linear spaces. Denote by $\mathcal{Q}$ the set of all absolutely convex $\sigma(F,E)$ compact subsets of F. Then the finest topology of a Schwartz space on E consistent with the duality $\langle E, F\rangle$ is the topology of uniform convergence on all those $A \in \in \mathcal{Q}$ for which there is $B \in \mathcal{Q}$ such that the topology of the normed space F_B and the topology $\sigma(F,E)$ coincide on A.

<u>Proof</u>. Let $\tau = \tau(E,F)$ be the Mackey topology on E consistent with the duality $\langle E, F\rangle$. In view of [3, prop. 3] it is sufficient to show that for every $A \in \in \mathcal{Q}$, A^0 is $\tau-(S)$-neighbourhood in E if and only if there is $B \in \mathcal{Q}$ such that the topology $\sigma(F,E)$ and the topology of the normed space F_B

coincide on λ . This is again easily seen by Proposition 1 and by the observation that for the neighbourhoods $\mathcal{V}$, $\mathcal{U}$, where $\mathcal{V} \subset \mathcal{U}$, in a topological linear space E , the following is equivalent:

a) The operator $E(\mathcal{V}, \mathcal{U}): E_{\mathcal{V}} \rightarrow E_{\mathcal{U}}$ is completely continuous.

b) The topology $\sigma(E', E)$ and the topology of the normed space $E_{\mathcal{V}^o}$ coincide on $\mathcal{U}^o$.

This completes the proof.

References

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