Commentationes Mathematicae Universitatis Carolinae

12,3 (1971)

THE EXISTENCE OF UPPER SEMICOMPLEMENTS IN LATTICES OF PRIMITIVE CLASSES

Jaroslav JEŽEK, Praha

Consider a type Δ of universal algebras, containing at least one at least binary function symbol. A.D. Bolbot [1] asks: is the variety of all Δ -algebras generated by a finite number of its proper subvarieties? It follows from Theorem 1 below that the answer is positive.

Results of [1] are essentially stronger than Theorems 3 and 4 of my paper [3].

§§ 1 and 2 contain some auxiliary definitions and lemmas. § 3 brings the main result. In § 4 we prove four rather trivial theorems that give some more information. Theorem 5 states that the answer to Bolbot's question is negative, if minimal subvarieties are considered instead of proper subvarieties.

§ 1. E -proofs, reduced length and (x, Δ) -equations

For the terminology and notation see § 1 of [2].

Let a type $\Delta = (m_i)_{i \in I}$ be fixed throughout this paper.

AMS, Primary 08A15, 08A25 Ref. Ž. 2.725.2
Secondary -

In auxiliary considerations we shall often make use of finite sequences. The sequence formed by $t_1, \ldots, t_n$ will be denoted by $\ulcorner t_1, \ldots, t_n \urcorner$. The case $n = 0$ is not excluded; the empty sequence is denoted by $\emptyset$. If $\sigma = \ulcorner t_1, \ldots, t_n \urcorner$ and $\varphi = \ulcorner u_1, \ldots, u_m \urcorner$ are two finite sequences, then $\ulcorner t_1, \ldots, t_n, u_1, \ldots, u_m \urcorner$ is denoted by $\sigma \odot \varphi$. Evidently, $\sigma \odot \emptyset = \emptyset \odot \sigma = \sigma$. If σ is given, then we define $\sigma^{[1]}, \sigma^{[2]}, \sigma^{[3]}, \ldots$ in this way: $\sigma^{[1]} = \sigma$; $\sigma^{[n+1]} = \sigma \odot \sigma^{[n]}$.

If a Δ -theory E (i.e. a set of Δ -equations, i.e. $E \subseteq W_\Delta \times W_\Delta$) is given, then for every $t \in W_\Delta$ we denote by $LC_E(t)$ the subset of W_Δ defined in this way: $u \in LC_E(t)$ if and only if there exists an endomorphism φ of W_Δ and an equation $\langle a, b \rangle \in E$ such that $\varphi(a) = t$ and $\varphi(b) = u$. Elements of $LC_E(t)$ are called leap-consequences of t by means of E .

If E is given, then we define a subset $IC_E(t)$ of W_Δ for every $t \in W_\Delta$ in this way: if either $t \in X$ or $t = f_i$ for some $i \in I$, $n_i = 0$, then $IC_E(t) = LC_E(t)$; if $t = f_i(t_1, \ldots, t_{n_i})$ where $n_i \geq 1$, then $IC_E(t) = LC_E(t) \cup \bigcup_{j=1}^{n_i} \{ f_i(t_1, \ldots, t_{j-1}, \xi, t_{j+1}, \ldots, t_{n_i});$ $\xi \in IC_E(t_j) \}$. Elements of $IC_E(t)$ are called immediate consequences of t by means of E .

By an E -proof we mean a finite, non-empty sequence $\ulcorner t_1, \ldots, t_n \urcorner$ of elements of W_Δ such that for every $j = 1, \ldots, n-1$ one of the following three cases takes place: either $t_j = t_{j+1}$ or t_j is an immediate consequence of t_{j+1} by means of E or t_{j+1} is an immediate consequence of t_j by means of E . A natural number j $(1 \leq j \leq n-1)$ is called leap in an E-proof $\ulcorner t_1, \dots, t_n \urcorner$ if either $t_j \in LC_E(t_{j+1})$ or $t_{j+1} \in LC_E(t_j)$. If u and v are two elements of W_Δ , then E-proofs $\ulcorner t_1, \dots, t_n \urcorner$ such that $t_1 = u$ and $t_n = v$ are called E-proofs of v from u. It is easy to prove that whenever E is a Δ-theory and $u, v \in W_\Delta$, then $E \vdash \langle u, v \rangle$ if and only if there exists an E-proof of v from u. An E-proof $\ulcorner t_1, \dots, t_n \urcorner$ is called minimal if every E-proof of t_n from t_1 has at least n members. If e is a Δ-equation, then $\{e\}$-proofs are called e-proofs.

Lemma 1. Let $h \in I$, $n_h \geq 2$; let $t, u \in W_\Delta$; put $a = f_h(t, u, t, t, \dots, t)$ and $b = f_h(u, t, t, t, \dots, t)$. Then every minimal $\langle a, b \rangle$-proof has at most one leap.

Proof. Let $\ulcorner t_1, \dots, t_n \urcorner$ be a minimal $\langle a, b \rangle$-proof; suppose that it has at least two leaps. Evidently, this proof has two leaps j, k $(1 \leq j < k \leq n-1)$ such that between them there are no leaps. There exists an endomorphism φ of W_Δ such that either

$$t_j = f_h(\varphi(t), \varphi(u), \varphi(t), \dots, \varphi(t)) \ \& \ t_{j+1} =$$
$$= f_h(\varphi(u), \varphi(t), \varphi(t), \dots, \varphi(t))$$

or $$t_j = f_h(\varphi(u), \varphi(t), \varphi(t), \dots, \varphi(t)) \ \& \ t_{j+1} =$$
$$= f_h(\varphi(t), \varphi(u), \varphi(t), \dots, \varphi(t)) .$$

There exists an endomorphism ψ of W_Δ such that either

$$t_k = f_h(\psi(t), \psi(u), \psi(t), \dots, \psi(t)) \ \& \ t_{k+1} =$$

$= f_k(\psi(u), \psi(t), \psi(t), \ldots, \psi(t))$

or on the contrary. If $k = j+1$, then evidently $t_j = t_{k+1}$ in all cases, so that $\ulcorner t_1, \ldots, t_j, t_{k+2}, \ldots, t_n \urcorner$ is a shorter $\langle a, b \rangle$-proof of t_n from t_1, a contradiction. Hence $k > j+1$. For every ℓ $(j \leq \ell \leq k+1)$ there evidently exist $w_{1,\ell}, \ldots, w_{n_k,\ell}$ such that $t_\ell = f_k(w_{1,\ell}, \ldots, w_{n_k,\ell})$. In all cases

$$\ulcorner t_1, \ldots, t_j, f_k(w_{2,j+2}, w_{1,j+2}, w_{3,j+2}, \ldots, w_{n_k,j+2}), \ldots, f_k(w_{2,k}, w_{1,k}, w_{3,k}, \ldots, w_{n_k,k}), t_{k+2}, \ldots, t_n \urcorner$$

is evidently a shorter $\langle a, b \rangle$-proof of t_n from t_1, a contradiction.

Let us assign to each $t \in W_\Delta$ a natural number $\ell(t)$, called the reduced length of t, in this way: if either $t \in X$ or $t = f_i$ for some $i \in I$, $n_i = 0$, then $\ell(t_i) = 1$; if $t = f_i(t_1, \ldots, t_{n_i})$ where $n_i \geq 1$, then $\ell(t) = \ell(t_1) + \ldots + \ell(t_{n_i})$.

Let a variable x be given. Denote by $T_\Delta(x)$ the set of all $t \in W_\Delta$ such that no f_i (where $n_i = 0$) and no variable different from x belongs to $S(t)$. ($S(t)$ is the set of all subwords of t.)

Δ-equations $\langle a, b \rangle$ such that both a and b belong to $T_\Delta(x)$ are called (x, Δ)-equations. The set of all (x, Δ)-equations $\langle a, b \rangle$ satisfying $\ell(a) = \ell(b)$ is denoted by $E_\Delta(x)$.

Lemma 2. Let $x \in X$ and $t \in T_\Delta(x)$. Then

$\ell(\varphi(t)) = \ell(t)\cdot\ell(\varphi(x))$ for every endomorphism φ of W_Δ.

Proof is easy (by the induction on t).

Lemma 3. Let a variable x, a Δ-theory $E \subseteq E_\Delta(x)$ and two elements u, v of W_Δ such that $E \vdash \langle u, v\rangle$ be given. Then $\ell(u) = \ell(v)$.

Proof. Applying Lemma 2, it is easy to prove the following assertion by the induction on a: whenever $a \in W_\Delta$ and $b \in \mathrm{IC}_E(a)$, then $\ell(a) = \ell(b)$.

§ 2. Occurrences of subwords; h-numbers

Let us call a subset A of W_Δ admissible if whenever $u, v \in A$ and $u \neq v$, then u is not a subword of v. Let an admissible set A be given. Then we assign to every $t \in W_\Delta$ a finite sequence $OCC_A(t)$ of elements of W_Δ in this way: if either $t \in X$ or $t = f_i$ for some $i \in I$, $n_i = 0$, then $OCC_A(t) = \ulcorner t \urcorner$ in the case $t \in A$ and $OCC_A(t) = \emptyset$ in the case $t \notin A$; if $t = f_i(t_1, \ldots, t_{n_i})$ where $n_i \geq 1$, then $OCC_A(t) = \ulcorner t \urcorner$ in the case $t \in A$ and $OCC_A(t) = OCC_A(t_1) \circ \ldots \circ OCC_A(t_{n_i})$ in the case $t \notin A$. Evidently, $OCC_A(t)$ is a finite sequence of elements, each of which belongs to A and is a subword of t; an element of A occurs in $OCC_A(t)$ if and only if it is a subword of t.

Let two natural numbers n, m be given, $n \geq 2$. Let $h \in I$, $n_h \geq 2$. Then $h_m^{n,1}$ ($h_m^{n,2}$, respectively) denotes the set of all $t = f_h(\alpha_1, \ldots, \alpha_{n_h}) \in W_\Delta$

such that $\ell(\alpha_1) = \ell(\alpha_3) = \ldots = \ell(\alpha_{n_h})$ & $\ell(\alpha_2) = n \cdot \ell(\alpha_1)$ ($\ell(\alpha_2) = \ell(\alpha_3) = \ldots = \ell(\alpha_{n_h})$ & $\ell(\alpha_1) = n \cdot \ell(\alpha_2)$, resp.) and $\ell(t) = m$. Evidently, the sets $h_m^{n,1}$ and $h_m^{n,2}$ are disjoint; put $h_m^n = h_m^{n,1} \cup h_m^{n,2}$. Let us call two elements of h_m^n similar if either they both belong to $h_m^{n,1}$ or they both belong to $h_m^{n,2}$. If $\sigma = \ulcorner t_1, \ldots, t_k \urcorner$ and $\varphi = \ulcorner u_1, \ldots, u_\ell \urcorner$ are two finite sequences of elements of h_m^n, then we write $\sigma \approx \varphi$ if and only if $k = \ell$ and t_j and u_j are similar for every $j = 1, \ldots, k$. Evidently, h_m^n is an admissible set.

Let an element $h \in I$ such that $n_h \geq 2$ be given; let $t \in W_\Delta$. By an h-number of t we mean any natural number $n \geq 2$ such that no element of $h_1^n \cup$ $\cup h_2^n \cup h_3^n \cup \ldots$ is a subword of t. Evidently, the set of all natural numbers that are not h-numbers of a given element $t \in W_\Delta$ is finite. By an h-number of a Δ-theory E we mean any natural number $n \geq$ ≥ 2 such that, for every $\langle a, b \rangle \in E$, n is an h-number of both a and b.

Lemma 4. Let $h \in I$, $n_h \geq 2$. Let E be a finite Δ-theory. The set of all natural numbers that are not h-numbers of E is finite.

Proof is evident.

If a variable x and an element $h \in I$ such that $n_h \geq 2$ is given, then we define elements $x^{1,h}$, $x^{2,h}$, $x^{3,h}, \ldots$ of W_Δ in this way: $x^{1,h} = x$; $x^{n+1,h} =$ $= f_h(x^{n,h}, \ldots, x^{n,h})$.

Lemma 5. Let $h \in I$, $n_h \geq 2$. Let $n \geq 2$ be a natural number, $x \in X$ and $u, v \in W_\Delta$; let $\langle f_h(x, x^{n,h}, x, \ldots, x), f_h(x^{n,h}, x, x, \ldots, x)\rangle \vdash \langle u, v\rangle$. Put $n^* = \ell(x^{n,h})$. Then

(i) for every natural number m the sequences $OCC_{h^{n^*}_m}(u)$ and $OCC_{h^{n^*}_m}(v)$ have an equal number of members;

(ii) if $u \neq v$, then there exists a natural number k such that $OCC_{h^{n^*}_k}(u) \approx OCC_{h^{n^*}_k}(v)$ does not hold.

Proof. We shall write OCC_m instead of $OCC_{h^{n^*}_m}$, as h and n^* are fixed here. Put $e = \langle f_h(x, x^{n,h}, x, \ldots, x), f_h(x^{n,h}, x, x, \ldots, x)\rangle$. We shall prove by the induction on u that whenever v is an element of W_Δ such that $e \vdash \langle u, v\rangle$, then (i) and (ii) take place. If either $u \in X$ or $u = f_i$ for some $i \in I$, $n_i = 0$, then $v = u$ and everything is evident. Let $u = f_i(u_1, \ldots, u_{n_i})$, where $n_i \geq 1$. By Lemma 1, it is sufficient to consider the following two cases:

Case 1: Some e-proof of v from u contains no leap. Then there evidently exist $v_1, \ldots, v_{n_i}$ such that $v = f_i(v_1, \ldots, v_{n_i})$ and $e \vdash \langle u_1, v_1\rangle, \ldots, e \vdash \langle u_{n_i}, v_{n_i}\rangle$. By Lemma 3 we have $\ell(u) = \ell(v)$, $\ell(u_1) = \ell(v_1), \ldots, \ell(u_{n_i}) = \ell(v_{n_i})$. Let us prove (i). If $m > \ell(u)$, then $OCC_m(u)$ and $OCC_m(v)$ are both empty; if $m < \ell(u)$, then the assertion follows from the induction hypothesis; it remains to consider the case $m = \ell(u)$. If $n_i = 1$, then $OCC_m(u) = OCC_m(u_1)$ and $OCC_m(v) = OCC_m(v_1)$, so that the assertion follows from the induction hypothesis. If $n_i \geq 2$, then $OCC_m(u)$ is either empty or equal to $\ulcorner u \urcorner$ and similarly for $OCC_m(v)$; if one of the elements u and v belongs to h_m^{n*}, then from $\ell(u_1) = \ell(v_1), \dots, \ell(u_{n_i}) = \ell(v_{n_i})$ it follows that the other belongs to h_m^{n*}, too. (i) is thus proved. Let us prove (ii). If $u \neq v$, then $u_j \neq v_j$ for some j $(1 \leq j \leq n_i)$; by the induction hypothesis there exists a number k such that $OCC_k(u_j) \approx OCC_k(v_j)$ does not hold. We have $u \notin h_k^{n*}$, because otherwise $n_i = n_k \geq 2$ and simultaneously $\ell(u) = k \leq \ell(u_j)$ would take place. Similarly $v \notin h_k^{n*}$. From this and from the fact that by the induction hypothesis (i) holds for $u_1, \dots, u_{n_i}$, we get that $OCC_k(u) \approx OCC_k(v)$ does not hold.

Case 2: Some e-proof of v from u contains exactly one leap. Then evidently $i = h$ and there exist $v_1, \dots, v_{n_h}$ such that $v = f_h(v_1, \dots, v_{n_h})$ and $e \vdash \langle u_1, v_2 \rangle$, $e \vdash \langle u_2, v_1 \rangle$, $e \vdash \langle u_3, v_3 \rangle, \dots, e \vdash \langle u_{n_h}, v_{n_h} \rangle$. Let us prove (i). If $m > \ell(u)$, then $OCC_m(u)$ and $OCC_m(v)$ are both empty; if $m = \ell(u)$, then $OCC_m(u) = \ulcorner u \urcorner$ and $OCC_m(v) = \ulcorner v \urcorner$; if $m < \ell(u)$, then the assertion follows from the induction hypothesis. For the proof of (ii) it is sufficient to put $k = \ell(u)$; we have evidently $OCC_k(u) = \ulcorner u \urcorner$ and

$OCC_{h}(v) = \ulcorner v \urcorner$; $\ulcorner u \urcorner \approx \ulcorner v \urcorner$ does not hold.

Lemma 6. Let $h \in I$, $n_h \geq 2$. Let a variable x, an element $t \in T_\Delta(x)$, an h-number m of t and an endomorphism φ of W_Δ be given. If some $w \in h_1^m \cup h_2^m \cup h_3^m \cup \ldots$ is a subword of $\varphi(t)$, then it is a subword of $\varphi(x)$.

Proof (by induction on t). The case $t = x$ is evident. Let $t = f_i(t_1, \ldots, t_{n_i})$ where $n_i \geq 1$. Let $w = f_h(\alpha_1, \ldots, \alpha_{n_h}) \in h_m^n$ be a subword of $\varphi(t)$. We have $w \neq \varphi(t)$, as $w = \varphi(t) = f_i(\varphi(t_1), \ldots, \varphi(t_{n_i}))$ would imply $i = h$ and $\alpha_1 = \varphi(t_1), \ldots, \alpha_{n_h} = \varphi(t_{n_h})$, so that by Lemma 2 easily $t \in h_{\ell(t)}^m$, a contradiction. Consequently, w is a subword of $\varphi(t_j)$ for some j $(1 \leq j \leq n_i)$; by the induction hypothesis (we may apply it, because m is an h-number of t_j, as well), w is a subword of $\varphi(x)$.

Lemma 7. Let $h \in I$, $n_h \geq 2$. Let a variable x, an element $t \in T_\Delta(x)$, a natural number $m \leq \ell(\varphi(x))$ and an endomorphism φ of W_Δ be given. Then $OCC_{h_m^n}(\varphi(t)) = (OCC_{h_m^n}(\varphi(x)))^{[\ell(t)]}$ for every $n \geq 2$.

Proof (by induction on t). The case $t = x$ is evident. Let $t = f_i(t_1, \ldots, t_{n_i})$ where $n_i \geq 1$. Write OCC instead of $OCC_{h_m^n}$. If $n_i \geq 2$, then we get $\varphi(t) \notin h_m^n$ from $m \leq \ell(\varphi(x))$; hence,

$$OCC\,\varphi(t) = OCC\,\varphi(t_1) \circ \ldots \circ OCC\,\varphi(t_{n_i}) =$$
$$= (OCC\,\varphi(x))^{[\ell(t_1)]} \circ \ldots \circ (OCC\,\varphi(x))^{[\ell(t_{n_i})]} = (OCC\,\varphi(x))^{[\ell(t)]}.$$

If $n_i = 1$, then $OCC\,\varphi(t) = OCC\,\varphi(t_1) =$

$= (OCC\ \varphi(x))^{[\ell(t_1)]} = (OCC\ \varphi(x))^{[\ell(t)]}$.

Lemma 8. Let $h \in I$, $n_h \geq 2$. Let $x \in X$, $u \in W_\Delta$ and $\langle a, b \rangle \in E_\Delta(x)$; let n be an h-number of both a and b . Then the following holds: whenever some v is an immediate consequence of u by means of $\langle a, b \rangle$, then $OCC_{h_m^n}(u) \approx OCC_{h_m^n}(v)$ for every m .

Proof (by induction on u). Write OCC instead of $OCC_{h_m^n}$. If either $u \in X$ or $u = f_i$ for some $i \in I$, $n_i = 0$, then either $v = u$ or there exists a finite sequence $i_1, \ldots, i_k$ of elements of I such that $n_{i_1} = \ldots = n_{i_k} = 1$ and $v = f_{i_1}(f_{i_2}(\ldots f_{i_k}(u)\ldots))$; evidently, in all cases the sequences $OCC(u)$ and $OCC(v)$ are both empty. Let $u = f_i(u_1, \ldots, u_{n_i})$ where $n_i \geq 1$.

Let firstly there exist a j $(1 \leq j \leq n_i)$ and a $v_j \in W_\Delta$ such that $v = f_i(u_1, \ldots, u_{j-1}, v_j, u_{j+1}, \ldots, u_{n_i})$ where v_j is an immediate consequence of u_j by means of $\langle a, b \rangle$. By Lemma 3 we have $\ell(u_j) = \ell(v_j)$. If $m > \ell(u)$, then $OCC(u)$ and $OCC(v)$ are both empty. If $m < \ell(u)$, then the assertion follows from the induction hypothesis. Let $m = \ell(u)$. If $n_i = 1$, then $OCC(u) = OCC(u_1)$ and $OCC(v) = OCC(v_1)$, so that the assertion follows from the induction hypothesis. If $n_i \geq 2$, then $OCC(u)$ is either empty or equal to $\ulcorner u \urcorner$, and similarly for $OCC(v)$, so that from $\ell(u_j) = \ell(v_j)$ we get easily $OCC(u) \approx OCC(v)$.

Let secondly there exist an endomorphism φ of

W_Δ such that $u = \varphi(a)$ and $v = \varphi(b)$. In this case we prove $OCC(u) = OCC(v)$. Suppose on the contrary that this does not hold. Evidently, some element of h_m^n is a subword of either u or v. By Lemma 6 we have $m \leq \ell(\varphi(x))$ and by Lemma 7 we get $OCC(\varphi(a)) = OCC(\varphi(b))$.

§ 3. The existence of upper semicomplements

Let us denote by ι_Δ the greatest and by ν_Δ the smallest element of $\mathcal{L}_\Delta$. If a and b are two elements of $\mathcal{L}_\Delta$, then their supremum in $\mathcal{L}_\Delta$ is denoted by $a \vee_\Delta b$ and their infimum by $a \wedge_\Delta b$. An element a of $\mathcal{L}_\Delta$ is called upper semicomplement in $\mathcal{L}_\Delta$ if there exists a $b \in \mathcal{L}_\Delta$ such that $b \neq \iota_\Delta$ and $a \vee_\Delta b = \iota_\Delta$.

To each Δ-theory E there corresponds an element in $\mathcal{L}_\Delta$; this element was denoted by $Cn(E)$ in [2].

Theorem 1. Let Δ be a type such that $n_h \geq 2$ for some $h \in I$. Let x be a variable and E a finite set of (x, Δ)-equations such that whenever $\langle a, b \rangle \in E$, then $\ell(a) = \ell(b)$. Then $Cn(E)$ is an upper semicomplement in $\mathcal{L}_\Delta$.

Proof. By Lemma 4 there exists a natural number $n \geq 2$ such that the number $m^* = \ell(x^{n,h})$ is an h-number of E. Put $e = \langle f_h(x, x^{n,h}, x, x, \dots, x)$, $f_h(x^{n,h}, x, x, x, \dots, x)\rangle$. It is sufficient to prove

$Cn(E) \vee_\Delta Cn(e) = \iota_\Delta$. Suppose on the contrary that there exists a Δ-equation $\langle u, v \rangle$ such that $u \neq v$, $E \vdash \langle u, v \rangle$ and $e \vdash \langle u, v \rangle$. By Lemma 5 there exists a natural number k such that $OCC_{h_k^{n*}}(u) \approx OCC_{h_k^{n*}}(v)$ does not hold. Lemma 8 implies $OCC_{h_k^{n*}}(u) \approx OCC_{h_k^{n*}}(v)$, a contradiction.

Remark. Let again Δ be such that $n_h \geq 2$ for some $h \in I$; let $x \in X$. By Theorem 1, $Cn(E)$ is an upper semicomplement in $\mathcal{L}_\Delta$ for every finite subset E of $E_\Delta(x)$. ($E_\Delta(x)$ is the set of all (x,Δ)-equations $\langle a, b \rangle$ such that $\ell(a) = \ell(b)$.) However, if $n_i \geq 1$ for all $i \in I$, then $Cn(E_\Delta(x))$ is not an upper semicomplement. This follows easily from Lemma 7 of [3].

§ 4. Some supplements

For every $t \in W_\Delta$ let $Var(t)$ be the set of all variables that are subwords of t. Let us denote by SL_Δ the set of all Δ-equations $\langle a, b \rangle$ satisfying $Var(a) = Var(b)$. It is easy to prove that SL_Δ is a fully invariant congruence relation of W_Δ, so that $SL_\Delta \in \mathcal{L}_\Delta$. Evidently, $SL_\Delta \neq \nu_\Delta$.

Theorem 2. For every type Δ, whenever E is an upper semicomplement in $\mathcal{L}_\Delta$, then $SL_\Delta \leq_\Delta E$, i.e. $E \subseteq SL_\Delta$.

Proof. Suppose on the contrary that there exists an equation $\langle a, b \rangle \in E$ such that $Var(a) \neq Var(b)$; let e.g. $Var(a) \nsubseteq Var(b)$; choose a variable

$x \in Var(a) \setminus Var(b)$. As E is an upper semicomplement, there exists an equation $\langle c, d \rangle$ such that $c \neq d$ and $Cn(\langle a, b \rangle) \vee_\Delta Cn(\langle c, d \rangle) = \iota_\Delta$. There exists a unique endomorphism φ of W_Δ such that $\varphi(x) = c$ for all $x \in X$; there exists a unique endomorphism ψ of W_Δ such that $\varphi(x) = d$ and $\varphi(x) = c$ for all $x \in X \setminus \{x\}$. We have evidently $\langle a, b \rangle \vdash \langle \varphi(a), \psi(a) \rangle$, $\langle c, d \rangle \vdash \langle \varphi(a), \psi(a) \rangle$ and $\varphi(a) \neq \psi(a)$, a contradiction.

Theorem 3. Let Δ be arbitrary. If a and b are two elements of $\mathcal{L}_\Delta$ such that $a \vee_\Delta b = \iota_\Delta$ and $a \wedge_\Delta b = \nu_\Delta$, then one of them is equal to ι_Δ and the other is equal to ν_Δ.

Proof follows from Theorem 2.

Theorem 4. Let Δ be arbitrary. If $a_1, \ldots, a_m$ $(m \geq 1)$ are elements of $\mathcal{L}_\Delta$ such that $a_1 \vee_\Delta \ldots \vee_\Delta a_m$ is an upper semicomplement in $\mathcal{L}_\Delta$, then at least one of them is an upper semicomplement in $\mathcal{L}_\Delta$.

Proof is trivial; the corresponding assertion holds in all lattices.

Theorem 5. Let Δ be such that $m_i \geq 1$ for some $i \in I$. Let $a_1, \ldots, a_m$ $(m \geq 1)$ be atoms in $\mathcal{L}_\Delta$. Then $a_1 \vee_\Delta \ldots \vee_\Delta a_m$ is not an upper semicomplement in $\mathcal{L}_\Delta$. Consequently, ι_Δ is not the supremum of a finite number of atoms in $\mathcal{L}_\Delta$.

Proof. By Theorem 4 it is enough to prove that no atom is an upper semicomplement. This follows from Theorem 3.

Remark. Bolbot [1] proved (for types Δ as in Theorem 1) that there exists a set A of atoms in $\mathcal{L}_\Delta$ such that ι_Δ is the supremum of A and $\mathrm{Card}\, A \leq \aleph_0 + \mathrm{Card}\, I$.

Problem. Consider, for example, only the most important case: I contains a single element i and $n_i = 2$. (Algebras of type Δ are just groupoids.) Find all Δ -equations e such that $Cn(e)$ is an upper semicomplement in $\mathcal{L}_\Delta$.

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Matematicko-fyzikální fakulta
Karlova universita
Sokolovská 83, Praha 8
Československo

(Oblatum 5.2.1971)