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ERROR - ESTIMATES FOR THE RITZ'S METHOD OF FINDING EIGENVALUES AND EIGENFUNCTIONS

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In [1] - [3] , we studied the method of least squares for approximating the eigenvalues and the eigenfunctions of a DS-operator. In this paper, we present a priori error-estimates for the Ritz's method for eigenvalue problems. Upper and lower error bounds are found.

We assume throughout this paper that A is a DS-operator whose domain $\mathcal{D}(A)$ is dense in a separable Hilbert space H , i.e., A is a symmetric operator in H such that the set of its eigenvalues is of the first category on the real axis and the spectrum $\mathcal{G}(A)$ is the closure of this set.

Suppose A is bounded below and such that the eigenvalues $\{\lambda_i\}$ of A satisfy the relations

(1) $$\lambda_1 < \lambda_2 < \ldots < \lambda_j < \ldots \ .$$

Let H_j be the closure of linear manifold generated by the eigenfunctions of A associated with the eigenva-

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lue λ_i. The symbol $P_i u$ will be used to denote the orthogonal projection u on H_i. We introduce the operator $B = (A - \mu I)^{\frac{1}{2}}$, where I denotes the identity operator in H and μ is a real number such that $\mu < \lambda_1$. It is evident that $\mu_i = \sqrt{\lambda_i - \mu}$, $i = 1, 2, \ldots$ are the eigenvalues of B and H_i is the closure of the linear manifold generated by the eigenfunctions of B associated with the eigenvalue μ_i. Therefore B is DS-operator. We remark that

$$\mathcal{D}(B) = \{ u \in H \, / \sum_{i=1}^{\infty} \mu_i^2 \, \| P_i u \|^2 < \infty \} \supset \mathcal{D}(A)$$

and $B u = \sum_{i=1}^{\infty} \mu_i P_i u$ for each $u \in \mathcal{D}(B)$.

Let $\{ \Psi_i \}_{i=1}^{\infty}$ be a B-complete system (cf. [6], [11]) and let R_m and $\mathcal{R}_m$ be subspaces of H determined by functions $\{ \Psi_i \}_{i=1}^m$ and $\{ B \Psi_i \}_{i=1}^m$, respectively. Denote

$$Q_m^{(m)} = \inf_{\substack{u \in R_m \cap H_m^{(2)} \\ \|u\| = 1}} (A u - \mu u, u) \, ,$$

where $H_m^{(1)} = \sum_{i=1}^{m-1} \oplus H_i$, $H_m^{(2)} = H \ominus H_m^{(1)}$ for $m > 1$ and $H_1^{(1)} = \{ \theta \}$, $H_1^{(2)} = H$.

Then the sequence $\{ Q_m^{(m)} \}_{m=1}^{\infty}$ is monotonically decreasing and converging to $\lambda_m - \mu$ (cf. Theorem 5 of [11]).

Let φ_1 be a normalized eigenfunction corresponding to μ_1. Let us construct the sequence of numbers $\{ q_m \}_{m=1}^{\infty}$ such that $q_m = \min_{\substack{u \in R_m \\ \|u\| = 1}} \| B u \|$. By Lemma 3 and Remark 3 of [3] we have for $m \geq m_0$

(2)
$$q_m^2 - \mu_1^2 \leq C_1 \cdot \| \varphi_1 - {}^{(m)}\varphi_1 \|^2 \ ,$$

where $C_1 = (\lambda_1 - \mu) \cdot (1 - \| \varphi_1 - {}^{(m)}\varphi_1 \|)^{-2}$,

${}^{(m)}\varphi_1$ is the orthogonal projection of φ_1 on

$\mathcal{L} \{ B \Psi_i \}_{i=1}^m$ and m_0 is a positive integer such that $^{(m_0)}\varphi_1 \neq 0$ and $^{(m_0-1)}\varphi_1 = 0$. If μ_1 is a simple eigen-value of B , then

(3)
$$q_m^2 - \mu_1^2 \geq C_2 \cdot \| \varphi_1 - {}^{(m)}\varphi_1 \|^2 \ ,$$
$$C_2 = \mu_1^2 \cdot (\mu_2^2 - \mu_1^2) \cdot (\mu_2^2 + \mu_1^2)^{-1}$$

and there exists $\{ u_m \}_{m=1}^\infty$ such that the following con-ditions are satisfied:

(4)

   1) $u_m \in R_m$, $\| u_m \| = 1$,

   2) $\| B u_m \| = q_m$,

   3) $\lim\limits_{m \to \infty} u_m = \varphi_1$,

   4) $(u_m, \varphi_1) \geq 0$ for $m = 1, 2, \ldots$.

    The proof is similar to that of Theorem 1 in [3] and Theorem 3 in [2]. Further there exist constants C_3 and C_4 such that for $m \geq m_0$ we have

(5)
$$\mu_1 \cdot \| \varphi_1 - {}^{(m)}\varphi_1 \| \leq \| B u_m - B \varphi_1 \| \leq C_3 \cdot \| \varphi_1 - {}^{(m)}\varphi_1 \|$$
$$\| \varphi_1 - \varphi_1^{(m)} \| \leq \| u_m - \varphi_1 \| \leq C_4 \cdot \| \varphi_1 - {}^{(m)}\varphi_1 \| \ ,$$

where $\varphi_1^{(m)}$ is the orthogonal projection of φ_1 on R_m (cf. Theorem 2 of [3]).

1. In this section, we give upper and lower bounds
for $Q_m^{(1)} - \lambda_1$ and for $\| u_m - \varphi_1 \|_1$, where $\| u \|_1^2 =$
$= (A u - \mu \cdot u, u)$, respectively.

Since μ is smaller than λ_1, the bilinear form
defined by $(u, v)_1 = (A u - \mu u, v)$, u, v in $\mathfrak{D}(A)$,
is a scalar product. Denote by $\mathcal{H}_1$ the complete hull
of $\mathfrak{D}(A)$ with the norm $\| u \|_1 = \sqrt{(u, u)_1}$. Let $\{ \Psi_i \}_{i=1}^{\infty}$
be a complete system in $\mathcal{H}_1$ and let $\lambda_1^{(m)}$ be an ap-
proximation to λ_1 obtained by applying the Ritz's
method to the subspace $R_m = \mathcal{L} \{ \Psi_i \}_{i=1}^{m}$ of $\mathcal{H}_1$. It
follows from the definition of B, that $\lambda_1^{(m)} = Q_m^{(1)} + \mu =$
$= Q_m^2 + \mu$. If, in addition, λ_1 is a simple eigenva-
lue of A, then $\{ u_m \}_{m=1}^{\infty}$ has the following properties

$$1) \quad u_m \in R_m, \quad \| u_m \| = 1,$$

$$(6) \qquad 2) \quad \| u_m \|_1^2 = \lambda_1^{(m)} - \mu,$$

$$3) \quad (u_m, u_{m+1}) \geq 0.$$

Therefore u_m is an approximation to φ_1 obtained by
applying the Ritz's method to R_m.

Since

$$(7) \quad \| \varphi_1 - \overset{(m)}{\varphi_1} \|^2 = \inf_{u \in R_m} \| \varphi_1 - B u \|^2 = \frac{1}{\lambda_1 - \mu} \cdot \inf_{v \in R_m} \| \varphi_1 - v \|_1^2$$

and $\| B u_m - B \varphi_1 \| = \| u_m - \varphi_1 \|_1$,

the following theorem is a direct consequence of (2) - (7).

Theorem 1. Let A be a DS-operator which is bounded
below. Let $\lambda_1 < \lambda_2 < \cdots$ be an enumeration of its dis-

tinct eigenvalues increasing order of values and let

μ be such a number that $\mu < \lambda_1$. Let $\{\psi_i\}_{i=1}^{\infty}$
be a complete system in $\mathcal{H}_1$. Denote by $\lambda_1^{(n)}$ an approximation to λ_1 obtained by applying the Ritz's
method to subspace $R_m = \mathcal{L}\{\psi_i\}_{i=1}^{n}$ of $\mathcal{H}_1$. Then there
exists a positive number D which does not depend on
m such that for $n \geq m_0$

$$\lambda_1^{(n)} - \lambda_1 \leq D \cdot \inf_{\substack{\varphi \in H_1 \\ \|\varphi\|=1}} \left(\inf_{u \in R_m} \| \varphi - u \|_1^2 \right) \, ,$$

where H_1 is the closure of linear manifold generated
by the eigenfunctions of A associated with the eigen-
value λ_1 and

$$m_0 = \inf_{u \in H_1} \max \{ n \mid (u, v)_1 = 0, \ v \in R_m \} \, .$$

If, in addition, λ_1 is a simple eigenvalue of A ,
then there exist constants $D_1, D_2 \neq 0, D_3, D_4$ which
do not depend on n such that

$$D_2 \cdot (E_1^{(n)})^2 \leq \lambda_1^{(n)} - \lambda_1 \leq D_1 \cdot (E_1^{(n)})^2 \, ,$$

$$E_1^{(n)} \leq \| u_m - \varphi_1 \|_1 \leq D_3 \cdot E_1^{(n)} \, ,$$

$$\| u_m - \varphi_1 \| \leq D_4 \cdot E_1^{(n)} \, ,$$

where u_m is the Ritz's approximation to a normalized
eigenfunction φ_1 with the properties (6) and $E_1^{(n)}$ is
the error of the best approximation to φ_1 by functions
of R_m in the norm $\| u \|_1$, i.e. $E_1^{(n)} = \inf_{v \in R_m} \| \varphi_1 - v \|_1$.

Remark 1. It follows from Lemma 3, Theorems 1 and 2
of [3] and from (7) that Theorem 1 is also valid when we

replace D_i by D_i^* for $i = 1,2,3,4$, where

$$D_1^* = (1 - v_n)^{-2}, \quad v_n = (\lambda_1 - \mu)^{-\frac{1}{2}} \cdot E_1^{(n)} ,$$

$$D_2^* = (1 + 2 \cdot \frac{\lambda_1 - \mu}{\lambda_2 - \lambda_1})^{-1} ,$$

$$D_3^* = (D_1^* / D_2^*)^{\frac{1}{2}} ,$$

$$D_4^* = [\, 2 D_1^* / (\lambda_2 - \lambda_1)\,]^{\frac{1}{2}} .$$

Consequently $\lim\limits_{n \to \infty} D_1^* = 1$.

2. In this section, we derive upper and lower bounds for the errors of Ritz's approximation to λ_j and φ_j , $j > 1$. For simplicity we assume that

(8) $\lambda_1 < \lambda_2 < \ldots < \lambda_j < \lambda_{j+1}$ and $\lambda_i, i = 1, \ldots, j$ are simple eigenvalues of A .

Let φ_i be a normalized eigenfunction of A corresponding to the eigenvalue λ_i for $i = 1, \ldots, j$.

We now present a number of results which is useful to have on record for later use. First of all we consider the problem of approximating the eigenfunctions of B .

<u>Lemma 1</u>. With assumption (8), let $\{v_n\}_{n=1}^{\infty}$ be a sequence of normalized functions belonging to $\mathcal{D}(B)$ such that $\lim\limits_{n \to \infty} \| B v_n \| = \mu_j$ and $\lim\limits_{n \to \infty} (v_n, \varphi_i) = 0$ for $i = 1, \ldots, (j-1)$. Then there exists a convergent subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ such that its limit is an eigenfunction of B belonging to μ_j .

<u>Proof</u>: By direct computation it follows that

$$\| B v_m \|^2 - \mu_j^2 \ge (\mu_{j+1}^2 - \mu_j^2) \cdot \sum_{i=j+1}^{\infty} \| P_i v_m \|^2 +$$
$$+ \sum_{i=1}^{j-1} (\mu_i^2 - \mu_j^2) \cdot \| P_i v_m \|^2 .$$

Since $\lim_{m \to \infty} \| P_i v_m \| = \lim_{m \to \infty} | (v_m, \varphi_i) | = 0$ for $i = 1, \ldots, (j-1)$ and $\lim_{m \to \infty} \| B v_m \| = \mu_j$ we have $\lim_{m \to \infty} | (v_m, \varphi_j) | = 1$ and the remainder of this lemma may be proved in the same way as Lemma 2 in [2].

The following theorem is a generalization of Theorem 3 from [2] and gives information on the construction of the approximation to φ_j .

Theorem 2. With the assumptions of Theorem 1 and (8) construct the sequences $\{ u_m^{(i)} \}_{m=1}^{\infty}$, $i = 1, \ldots, j$ with the following properties

1) $q_m^{(i)} = \| B u_m^{(i)} \| = \min_{\substack{u \in R_m \\ (u, u_n^{(k)}) = 0, k = 1, \ldots, (i-1) \\ \| u \| = 1}}$

for $i = 2, \ldots, j$; $n \ge i$;

$$q_m^{(1)} = \| B u_m^{(1)} \| = \min_{u \in R_m, \| u \| = 1} \| B u \| ,$$

2) $\| u_m^{(i)} \| = 1, i = 1, \ldots, j$; $n \ge i$,

3) $(u_m^{(i)}, u_{m+1}^{(i)}) \ge 0$, $i = 1, \ldots, j$; $n \ge i$.

Then

a) The sequence $\{ u_m^{(i)} \}_{m=1}^{\infty}$ converges to a normalized eigenfunction φ_i of B associated with the eigenvalue $\mu_i = \sqrt{\lambda_i - \mu}$ for $i = 1, \ldots, j$,

b) $(B u_m^{(i)}, B u_m^{(k)}) = 0$ for $i \ne k$; $i, k = 1, \ldots, j$.

Proof: In case $j = 1$ the proof follows from Theorem 3 and Lemma 3 of [2]. We now proceed inductively. Let the theorem be true for $i = 1, \ldots, (j-1)$. It then follows, since $\lim\limits_{n \to \infty} \| u_n^{(i)} - \varphi_i \| = 0$ for $i = 1, \ldots, (j-1)$, that $\lim\limits_{n \to \infty} \alpha_n^{(i)} = 0$ for $i = 1, \ldots, (j-1)$, where

$$\alpha_n^{(i)} = (u_n^{(j)}, \varphi_i) = (u_n^{(j)}, \varphi_i - u_n^{(i)}) .$$

Now, if $j^{-1} > \varepsilon > 0$, there exists a positive integer m_0 such that for $m \geq m_0$ $\| B y \| \leq q_n^{(j)} + \varepsilon$, where

$$y = \frac{u_n^{(j)} - \sum\limits_{k=1}^{j-1} \alpha_n^{(k)} \varphi_k}{\| u_n^{(j)} - \sum\limits_{k=1}^{j-1} \alpha_n^{(k)} \varphi_k \|} .$$

Since $y \perp \mathscr{L} \{ \psi_i \}_{i=1}^{j-1}$, we have $\| B y \| \geq (Q_n^{(j)})^{\frac{1}{2}}$. Therefore

$$(9) \qquad \lim\limits_{n \to \infty} q_n^{(j)} \geq \lim\limits_{n \to \infty} (Q_n^{(j)})^{\frac{1}{2}} = \mu_j .$$

On the other hand, let us construct normalized functions $v_m \in R_m \cap H_j^{(2)}$, $m \geq j$ so that $\| B v_m \| = (Q_n^{(j)})^{\frac{1}{2}}$. Let $\beta_n^{(i)} = (v_n, u_m^{(i)}) = (v_n, u_m^{(i)} - \varphi_i)$.

Since $\lim\limits_{n \to \infty} \beta_n^{(i)} = 0$ for $i = 1, \ldots, (j-1)$, there exists a positive integer m_1 such that for $j^{-1} > \varepsilon > 0$

$$\| B w \| \leq (Q_n^{(j)})^{\frac{1}{2}} + \varepsilon \qquad \text{for } m \geq m_1 ,$$

where $w = v_n - \sum\limits_{i=1}^{j-1} \beta_n^{(i)} u_m^{(i)}$.

Since $w \perp \mathscr{L} \{ u_m^{(i)} \}_{i=1}^{(j-1)}$, it follows that $q_n^{(j)} \leq \| B w \|$ and hence

(10) $\lim\limits_{n \to \infty} q_n^{(j)} \leqq \lim\limits_{n \to \infty} (Q_n^{(j)})^{\frac{1}{2}} = \mu_j$.

Therefore, we see by (9) that $\lim\limits_{n \to \infty} q_n^{(j)} = \mu_j$.
Since $\lim\limits_{n \to \infty} (u_n^{(j)}, \varphi_i) = 0$ for $i = 1, ..., (j-1)$,
we know by Lemma 1 that the sequence $\{u_n^{(j)}\}_{m=1}^{\infty}$ con-
tains a convergent subsequence and that every its con-
vergent subsequence converges to a limit which is a nor-
malized eigenfunction of B associated with the eigen-
value μ_j . It follows that the sequence $\{u_n^{(j)}\}_{m=1}^{\infty}$
contains at most two accumulation points. These points
are φ_j and $(-\varphi_j)$, where φ_j is a normalized eigen-
function of B associated with the eigenvalue μ_j .The
remainder of the proof is similar to that in the proof
of Theorem 3 in [2].

The assumption 3) implies that $\{u_n^{(j)}\}_{m=1}^{\infty}$ has
one accumulation point. It then follows, by virtue of
Lemma 1, that the sequence $\{u_n^{(j)}\}_{m=1}^{\infty}$ is convergent
and so the first part of this theorem is proved.

To prove b), let $\{x_k\}_{k=1}^{n+1-i}$ be an orthonormal
basis of the space $X = \{u / u \in R_m, (u, u_n^{(\ell)}) = 0$
for $\ell = 1, ..., (i-1)\}$. Let $u_n^{(i)} = \sum\limits_{k=1}^{n+1-i} \alpha_k x_k$.
Using the definition of $u_n^{(i)}$, we have
$$\sum\limits_{\ell=1}^{n+1-i} \alpha_\ell [(Bx_k, Bx_\ell) - \delta_{\ell k} \cdot (q_n^{(i)})^2] = 0$$
for $k = 1, ..., (m+1-i)$
and hence
$$(Bu_n^{(i)}, Bx_k) = (q_n^{(i)})^2 \cdot \alpha_k, \quad k = 1, ..., (m+1-i).$$
By direct calculation we see that
$$(Bu_n^{(i)}, Bv) = (q_n^{(i)})^2 \cdot (u_n^{(i)}, v) \quad \text{for any } v \in X$$

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and this completes the proof of b). Theorem 2 is completely proved.

The following theorem states how accurately the eigenfunction φ_j and the eigenvalue μ_j of B can be approximated by functions of $\mathcal{L}\{\Psi_i\}_{i=1}^m$ for $m \geq \geq j$.

__Theorem 3.__ Under the same hypotheses as in Theorem 2 there exist a positive integer m_0 and the constants C_1, $C_2 \neq 0$, C_3, C_4 which are independent of m such that for $m \geq m_0$

(a) $C_2 \cdot \|\varphi_j - {}^{(m)}\varphi_j\|^2 \leq \varrho_m^{(j)} - \mu_j \leq C_1 \cdot \max\limits_{i=1,\dots,j} \|\varphi_i - {}^{(m)}\varphi_i\|^2$,

(b) $\mu_j \cdot \|\varphi_j - {}^{(m)}\varphi_j\| \leq \|B u_m^{(j)} - B\varphi_j\| \leq C_3 \cdot \max\limits_{i=1,\dots,j} \|\varphi_i - {}^{(m)}\varphi_i\|$,

(c) $\|u_m^{(j)} - \varphi_j\| \leq C_4 \cdot \max\limits_{i=1,\dots,j} \|\varphi_i - {}^{(m)}\varphi_i\|$,

where ${}^{(m)}\varphi_i$ is the orthogonal projection of φ_i on

$\mathcal{R}_m = \mathcal{L}\{B\Psi_k\}_{k=1}^m$ for $i = 1,\dots, j$.

__Proof:__ We proceed by induction. For $j = 1$ the statement follows from Theorems 1 and 2 of [3]. We now define T as the restriction of B to $\mathcal{R}_m = \mathcal{L}\{\Psi_k\}_{k=1}^m$. Since $0 \in \sigma(B)$, it follows that T and T^{-1} are continuous linear operators on $\mathcal{R}_m$ and $\mathcal{R}_m = \mathcal{L}\{B\Psi_k\}_{k=1}^m$, respectively. In a similar way, by methods analogous to those employed in the proof of Lemma 1 from [3], we can obtain

$$\|{}^{(m)}\varphi_j\|^2 - \mu_j^2 \cdot \|T^{-1}{}^{(m)}\varphi_j\|^2 \leq \|\varphi_j - {}^{(m)}\varphi_j\|^2$$

for each positive integer n .

Let

$$u = T^{-1 (n)}\varphi_j + \sum_{i=1}^{j-1} c_i u_n^{(i)} ,$$

where

$$c_i = - (T^{-1 (n)}\varphi_j , u_n^{(i)}) .$$

Then $(u, u_n^{(i)}) = 0$ for $i = 1,...,(j-1)$ and from Theorem 2 it follows that

$$\| u \|^2 = \| T^{-1 (n)}\varphi_j \|^2 - \sum_{i=1}^{j-1} c_i^2 .$$

It is easy to see that

$$c_i = - (T^{-1(n)}\varphi_j , u_n^{(i)} - \varphi_i) - \frac{1}{\mu_i} (^{(n)}\varphi_j , \varphi_i), \quad i = 1,...,(j-1).$$

Since

$$\| T^{-1 (n)}\varphi_j \| \leq \frac{1}{\mu_1} \cdot \| ^{(n)}\varphi_j \| \leq \frac{1}{\mu_1}$$

and $| (^{(n)}\varphi_j , \varphi_i) | \leq \| \varphi_j - ^{(n)}\varphi_j \|$ for $i = 1,...,(j-1)$,

we conclude by induction that

(11) $\qquad | c_i | \leq C \cdot \ell_j \qquad$ for $\quad i = 1,...,(j-1)$,

where $\ell_j = \max\limits_{i=1,...,j} \| \varphi_i - ^{(n)}\varphi_i \|$ and C is a constant which does not depend on n .

But $\lim\limits_{n \to \infty} \ell_j = 0$ and hence there exists for $0 < < \varepsilon < \mu_j^{-1}$ a positive integer n_0 such that $\| u \| \geq \geq \frac{1}{\mu_j} - \varepsilon > 0$.

By the definition of $q_{2m}^{(j)}$, we have

(12) $\qquad q_{2m}^{(j)} - \mu_j \leq C_m \cdot (\| B u \|^2 - \mu_j^2 \cdot \| u \|^2) ,$

where

$$C_m = \frac{1}{\| u \| \cdot (\| B u \| + \mu_j \cdot \| u \|)}$$

By direct calculation we see that

(13) $$\lim_{m \to \infty} C_m = \frac{1}{2}\mu_j \quad .$$

We have

(14) $$\|Bu\|^2 - \mu_j^2 \cdot \|u\|^2 = (\|{}^{(m)}\varphi_j\|^2 - \mu_j^2 \cdot \|T^{-1}\,{}^{(m)}\varphi_j\|^2) +$$
$$+ \sum_{i=1}^{j-1} c_i^2 \cdot [(q_m^{(i)})^2 + \mu_j^2] + 2 \cdot \sum_{i=1}^{j-1} c_i \cdot ({}^{(m)}\varphi_j , Bu_m^{(i)}),$$

by Theorem 2.

Further, by induction

(15) $$|({}^{(m)}\varphi_j , Bu_m^{(i)})| = |({}^{(m)}\varphi_j , Bu_m^{(i)} - B\varphi_i)| +$$
$$+ |({}^{(m)}\varphi_j - \varphi_j , B\varphi_i)| \leq D \cdot e_j$$

for $i = 1, \ldots, (j-1)$, where D is a constant which does not depend on m. On the basis of (11) - (15)

$$q_m^{(j)} - \mu_j \leq C_1 \cdot e_j^2 \quad \text{for} \quad m \geq m_0 \quad .$$

For proving (b) we remark that

(16) $$\|Bu_m^{(j)} - B\varphi_j\|^2 = (q_m^{(j)})^2 - \mu_j^2 + 2\mu_j^2 \cdot (1 - \alpha_j^{(j)}) ,$$

where $\alpha_j^{(j)} = (u_m^{(j)} , \varphi_j)$. By Theorem 2 it follows that $1 \geq \alpha_j^{(j)} \geq 0$, whence $1 - \alpha_j^{(j)} \leq 1 - (\alpha_j^{(j)})^2$.

Writing $u_m^{(j)} = \sum_{i=1}^{\infty} \alpha_i^{(j)} \cdot \varphi_i$, $\alpha_i^{(j)} = (u_m^{(j)} , \varphi_i)$, we have

$$(q_m^{(j)})^2 - \mu_j^2 \geq \mu_{j+1}^2 - \mu_j^2 - \sum_{i=1}^{j} (\alpha_i^{(j)})^2 \cdot (\mu_{j+1}^2 - \mu_i^2),$$

whence

(17) $$1 - (\alpha_j^{(j)})^2 \leq \frac{1}{\mu_{j+1}^2 - \mu_j^2} \cdot ((q_m^{(j)}) - \mu_j^2) +$$
$$+ \frac{\mu_{j+1}^2 - \mu_1^2}{\mu_{j+1}^2 - \mu_j^2} \cdot \sum_{i=1}^{j-1} (\alpha_i^{(j)})^2 \quad .$$

We now show that an estimate for $\sum\limits_{i=1}^{j-1}(\alpha_i^{(j)})^2$ leads to an estimate for $(q_m^{(j)})^2 - \mu_j^2$. Let us consider the matrix $\mathcal{A} = \{\alpha_m^{(i)}\}_{m,i=1,\ldots,(j-1)}$. By the induction we have $\lim\limits_{m\to\infty}\alpha_m^{(i)} = \sigma_{m,i}$. From this it follows that there exists for $0 < \varepsilon < 1$ a positive integer m_1 such that for $m \geq m_1$

(18) $|\mathcal{A}^{-1}| < 1 + \varepsilon$ and $|\mathcal{A}^{-1}\mu| \geq (1-\varepsilon)\cdot|\mu|$,

where $|\cdot|$ denotes the Euclidean norm.

Define for $m \geq m_1$

(19) $$\kappa = \mathcal{A}^{-1}\cdot\alpha^{(j)},$$

where $\alpha^{(j)} = \{\alpha_m^{(j)}\}_{m=1}^{j-1}$ and $\kappa = \{\kappa_i\}_{i=1}^{j-1}$.

We introduce $\eta_j^2 = \dfrac{1}{1+|\kappa|^2}$, $\eta_i = -\kappa_i\cdot\eta_j$ for $i = 1,\ldots,(j-1)$. Then

(20) $\sum\limits_{i=1}^{j}\eta_i^2 = 1$ and $\sum\limits_{i=1}^{j-1}\eta_i^2 = \dfrac{|\kappa|^2}{1+|\kappa|^2}$.

It follows from (18) - (20) that

(21) $\sum\limits_{i=1}^{j-1}\eta_i^2 \geq C\cdot\sum\limits_{i=1}^{j-1}(\alpha_i^{(j)})^2$ for $m \geq m_0$,

where $$C = \frac{(1-\varepsilon)^2}{1+(1+\varepsilon)^2} > 0.$$

Letting $v = \sum\limits_{i=1}^{j}\eta_i u_m^{(i)}$ we find $\|v\| = 1$ and $(v, \varphi_i) = 0$ for $i = 1,\ldots,(j-1)$. Hence, by the definition of $Q_m^{(j)}$,

$$\|Bv\|^2 \geq Q_m^{(j)} \geq \mu_j^2$$

and using Theorem 2 and (21) we find for $n \geq m_1$

$$(q_m^{(j)})^2 - \mu_j^2 \geq (q_m^{(j)})^2 - \| Bv \|^2 =$$

$$(22) \quad = \sum_{i=1}^{j-1} [(q_m^{(j)})^2 - (q_m^{(i)})^2] \cdot p_i^2 \geq D \cdot \sum_{i=1}^{j-1} (\alpha_i^{(j)})^2 \; ,$$

where D is a constant which does not depend on m.
It follows from (16),(17) and (22) that there exists a
constant $C > 0$ such that for $n \geq m_1$

$$(23) \quad \| B u_m^{(j)} - B \varphi_j \|^2 \leq C \cdot [(q_m^{(j)})^2 - \mu_j^2] \; .$$

This, together with (a), leads to the first assertion of
(b).

It is easy to see that

$$(24) \; \| B u_m^{(j)} - B \varphi_j \|^2 = \mu_j^2 \cdot \| \frac{1}{\mu_j} \cdot B u_m^{(j)} - \varphi_j \|^2 \geq \mu_j^2 \cdot \| \varphi_j - {}^{(n)} \varphi_j \|^2$$

and this completes the second assertion of (b).

Since $\| B u_m^{(j)} - B \varphi_j \| \geq \mu_1 \cdot \| u_m^{(j)} - \varphi_j \|$, the
right side of (c) follows at once from (b). The left si-
de of (a) follows from (23) and (24) and this completes
the induction and the proof of Theorem 3.

As a corollary to Theorems 2 and 3, we obtain the
main result of this paper.

<u>Theorem 4.</u> Let A be a DS-operator which is boun-
ded below. Let $\lambda_1 < \lambda_2 < \ldots$ be an enumeration of
its distinct eigenvalues increasing order of values and
let μ be such a number that $\mu < \lambda_1$. Suppose λ_i,
$i = 1, \ldots, j$ are simple. Denote by $(\mu, v)_1$ the
scalar product $(A\mu - \mu\mu, v)$. Let $\mathcal{H}_1$ be the
complete hull of $\mathcal{D}(A)$ with the norm $\| \mu \|_1$.

Let $\{\Psi_i\}_{i=1}^{\infty}$ be a complete system in $\mathcal{H}_1$. Construct the sequences $\{u_n^{(i)}\}_{n=1}^{\infty}$, $i = 1, \ldots, j$ with the following properties

1) $\min\limits_{\substack{u \in \mathcal{L}\{\Psi_k\}_{k=1}^{n} \\ \|u\| = 1}} (Au, u) = (Au_n^{(1)}, u_n^{(1)}) = \lambda_n^{(1)}$,

$\min\limits_{\substack{u \in \mathcal{L}\{\Psi_k\}_{k=1}^{n}}} (Au, u) = (Au_n^{(i)}, u_n^{(i)}) = \lambda_n^{(i)}, \quad i = 2, \ldots, j$,

$(u, u_n^{(k)}) = 0, \quad k = 1, \ldots, (i-1)$
$\|u\| = 1$

2) $\quad \|u_n^{(i)}\| = 1$

3) $\quad (u_n^{(i)}, u_{n+1}^{(i)}) \geqq 0$.

Then

a) The sequence $\{u_n^{(i)}\}_{n=1}^{\infty}$ converges to a normalized eigenfunction φ_i of A associated with the eigenvalue λ_i for $i = 1, \ldots, j$ and

b) Denote by $E_i^{(n)}$ the error of the best approximation to φ_i by functions of $\mathcal{L}\{\Psi_i\}_{i=1}^{n}$ in the norm $\|\cdot\|_1$, i.e.,

$$E_i^{(n)} = \inf_{v \in \mathcal{L}\{\Psi_i\}_{i=1}^{n}} \|\varphi_i - v\|_1 \quad .$$

There exist a positive integer n_0 and constants C_1, $C_2 \neq 0$, C_3, C_4 which do not depend on n such that for $n \geq n_0$

$$C_2 \cdot (E_j^{(n)})^2 \leq \lambda_n^{(j)} - \lambda_j \leq C_1 \cdot [\max_{i=1,\ldots,j} E_i^{(n)}]^2 \quad ,$$

$$E_j^{(n)} \leq \|u_n^{(j)} - \varphi_j\|_1 \leq C_3 \cdot \max_{i=1,\ldots,j} E_i^{(n)} \quad ,$$

$$\|u_n^{(j)} - \varphi_j\| \leq C_4 \cdot \max_{i=1,\ldots,j} E_i^{(n)} \quad .$$

Remark 2. The function $u_m^{(j)}$ and the number $\lambda_m^{(j)}$ in Theorem 4 are the Rayleigh-Ritz approximations to φ_j and λ_j, respectively.

Proof: Let $B = (A - \mu I)^{\frac{1}{2}}$. The proof of a) follows at once from Theorem 2. Since

$$E_i^{(m)} = \inf_{v \in \mathcal{L}\{\varphi_i\}_{i=1}^m} \| B(\varphi_i - v) \| = \sqrt{\lambda_i - \mu} \cdot \| \varphi_i - {}^{(m)}\varphi_i \|$$

and $\lambda_m^{(j)} - \lambda_j = (q_m^{(j)})^2 - \mu_j^2$ the assertion of b) follows from Theorem 3.

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