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ELLIPTIC POINTS IN ONE-DIMENSIONAL HARMONIC SPACES

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Introduction.

Let X be a locally compact space. By a harmonic sheaf on X we mean a map $\mathcal{H}$ assigning to each open set $\mathcal{U} \subset X$ a vector space $\mathcal{H}_{\mathcal{U}}$ (over the real number field) of finite real-valued continuous functions (called harmonic functions) on $\mathcal{U}$ in such a way that $(X, \mathcal{H})$ represents a harmonic space satisfying the axioms of N. Boboc, C. Constantinescu and A. Cornea, [2]. Let us recall that an open set $\mathcal{U} \subset X$ is termed regular provided it is relatively compact, its boundary $\partial \mathcal{U}$ is non-void and each continuous function f on $\partial \mathcal{U}$ possesses a unique continuous extension to $\overline{\mathcal{U}}$ (= the closure of $\mathcal{U}$) whose restriction $H_f^{\mathcal{U}}$ to $\mathcal{U}$ is harmonic and, in addition, non-negative on $\mathcal{U}$ whenever $f \geq 0$ on $\partial \mathcal{U}$. Given a regular set $\mathcal{U}$, then with each $x \in \mathcal{U}$ there is associated a Radon measure $\omega_x^{\mathcal{U}}$ (called harmonic measure) on $\partial \mathcal{U}$, which is defined by the map $\omega_x^{\mathcal{U}}: f \longrightarrow H_f^{\mathcal{U}}(x)$. Its support will be denoted by $\operatorname{spt} \omega_x^{\mathcal{U}}$. We shall say that $\mathcal{H}$ is elliptic at $x \in X$ (or that x is an

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elliptic point of the harmonic space $(X, \mathcal{H}))$ provided x possesses a fundamental system of regular neighborhoods $\mathcal{U}$ such that $\operatorname{spt} \omega_x^{\mathcal{U}} = \partial \mathcal{U}$. The set of all elliptic points will be denoted by $E(X, \mathcal{H})$. In general, little can be said about $E(X, \mathcal{H})$ and, actually, $E(X, \mathcal{H})$ may be empty as shown by the standard example where $X = R^{n+1}$ is the Euclidean $(n+1)$ -space and harmonic functions are solutions of the heat equation (cf. H. Bauer [1]). The present note centers around the investigation of $E(X, \mathcal{H})$ for the special case when X is a 1-manifold. The following results will be proved.

Theorem 1. If $\mathcal{H}$ is a harmonic sheaf on a 1-manifold X, then $E(X, \mathcal{H})$ is an open set everywhere dense in X and each component of $E(X, \mathcal{H})$ has a countable base.

Suppose now that X is a 1-manifold on which there has been fixed an orientation. This orientation induces a linear order on each arc C (= a subspace which is homeomorphic with the real line R^1) in X and for each $x \in C$ we may thus speak of the left-hand component and the right-hand component of $C \setminus \{x\}$ denoting them by $C^-(x)$ and $C^+(x)$, respectively. We shall denote by $F^+(X, \mathcal{H})$ the set of all $x \in X$ for which there is an arc C, $x \in C \subset X$, such that $\operatorname{spt} \omega_x^{\mathcal{U}} \subset C^+(x)$ whenever $\mathcal{U}$ is a regular set with $x \in \mathcal{U} \subset \overline{\mathcal{U}} \subset C$. Replacing $C^+(x)$ by $C^-(x)$ we define analogously the set

$F^-(X, \mathcal{H})$.

Theorem 2. $F^+(X, \mathcal{H})$, $F^-(X, \mathcal{H})$ are separated subsets of X and $F^+(X, \mathcal{H}) \cup$ $\cup F^-(X, \mathcal{H}) = X \setminus E(X, \mathcal{H})$.

Theorem 3. If X is an oriented 1-manifold and F^+, F^- are arbitrary separated subsets of X such that $E = X \setminus (F^+ \cup F^-)$ is an everywhere dense open set whose components have a countable base each, then there is always a harmonic sheaf $\mathcal{H}$ on X such that $F^+ = F^+(X, \mathcal{H})$, $F^- = F^-(X, \mathcal{H})$ and, consequently, $E = E(X, \mathcal{H})$.

Several related results will also be included and the structure of all absorbent sets will be described.

§ 1.

This paragraph includes several auxiliary results dealing with the case when X is an open interval in R^1 . If $a \leq b$ are elements of the extended real line, then we use the symbols

$\langle a, b) = \{x; x \in R^1, a \leq x < b\}$, $(a, b\rangle =$ $= \{x; x \in R^1, a < x \leq b\}$,

$\langle a, b\rangle = \langle a, b) \cup (a, b\rangle$, $(a, b) = \langle a, b) \cap (a, b\rangle$

to denote the intervals with end-points a, b. If Q is a compact set, then $\mathcal{C}(Q)$ will denote the space of all finite real-valued continuous functions on Q . Throughout this paragraph we assume that for each open set $\mathcal{U} \subset X$ there is given a vector space $\mathcal{H}_{\mathcal{U}}$ of real-valued continuous functions (called harmonic functions) on $\mathcal{U}$ such that the following axioms are satisfied:

(I) Sheaf axiom: If $\mathcal{U}_1 \subset \mathcal{U}_2$ are open sets, then

$$h \in \mathcal{H}_{\mathcal{U}_2} \Longrightarrow \mathrm{Rest}_{\mathcal{U}_1} h \in \mathcal{H}_{\mathcal{U}_1}$$

(where, as usual, $\mathrm{Rest}_{\mathcal{U}_1} h$ denotes the restriction of h to $\mathcal{U}_1$). If $\{\mathcal{U}_\lambda\}_{\lambda \in \Lambda}$ is a system of open sets and h is a function on $\mathcal{U} = \bigcup_{\lambda \in \Lambda} \mathcal{U}_\lambda$, then h is harmonic on $\mathcal{U}$ provided $\mathrm{Rest}_{\mathcal{U}_\lambda} h \in \in \mathcal{H}_{\mathcal{U}_\lambda}$ for each $\lambda \in \Lambda$.

(II) Basis axiom: Open sets that are regular (in the sense described in the introduction) form a base for the topology of X .

(III) Minimum principle: If $\langle a, b \rangle \subset X$ is a compact interval and $h \in \mathcal{C}(\langle a, b \rangle)$ is harmonic on (a, b), then $h \geq 0$ on $\langle a, b \rangle$ provided $h(a) \geq 0$ and $h(b) \geq 0$.

(IV) Harmonicity of constants: Constant functions are harmonic on X .

1.1. Remark. We shall denote by $\partial \mathcal{U}$ and $\overline{\mathcal{U}}$ the boundary and the closure of $\mathcal{U} \subset X$, respectively. Let $\mathcal{U}$ be a regular set. According to (IV), the harmonic measure $\omega_x^{\mathcal{U}}$ corresponding to $x \in \mathcal{U}$ is a probability measure. Each component of $\mathcal{U}$ is also regular and $\mathrm{spt}\, \omega_x^{\mathcal{U}}$ is contained in the boundary of that component of $\mathcal{U}$ which contains x (compare H. Bauer [1], the proofs of 1.3.4 and 1.5.1). Consequently, $\mathcal{U}$ is an interval provided $spt\ \omega_x^{\mathcal{U}} = \partial \mathcal{U}$ for some $x \in \mathcal{U}$. In view of (II), regular intervals form a base for the topology of X. In accordance with the introduction, we shall say that $x \in X$ is an elliptic point if x possesses a fundamental system of neighborhoods formed by regular intervals I such that $spt\ \omega_x^I = \partial I$. The set of all elliptic points will be denoted by E.

1.2. Lemma. Let $\langle a, b \rangle \subset X$ be a compact interval, $h \in \mathcal{C}(\langle a, b\rangle)$, $h(a) = 0$, $h(b) = 1$, $I = (a, b)$, $\mathrm{Rest}_I h \in \mathcal{H}_I$. Then there are $a', b' \in \langle a, b \rangle$ such that $a \leq a' < b' \leq b$, $h(\langle a, a' \rangle) = \{0\}$, $h(\langle b, b' \rangle) = \{1\}$ and h is (strictly) increasing on $\langle a', b' \rangle$. If, in addition, I is regular, then for any $x \in I$

(1) $$spt\ \omega_x^I = \{a, b\} \Longrightarrow a' < x < b' ,$$

(2) $$spt\ \omega_x^I = \{a\} \Longrightarrow a' \geq x ,$$

(3) $$spt\ \omega_x^I = \{b\} \Longrightarrow b' \leq x .$$

Proof. By (III), $h \geq 0$ on $\langle a, b \rangle$. Assuming $h(x) > h(y)$ for a couple of points $x < y$ in $\langle a, b \rangle$ and defining $\tilde{h}(t) = h(y) - h(t)$ for $t \in \langle a, y \rangle$, we obtain an $\tilde{h} \in \mathcal{C}(\langle a, y\rangle)$ with $\mathrm{Rest}_{(a,y)} \tilde{h} \in \mathcal{H}_{(a,y)}$, $\tilde{h}(a) \geq 0$, $\tilde{h}(y) = 0$, $\tilde{h}(x) < 0$. This contradiction shows that h must be non-decreasing on $\langle a, b \rangle$. Put $a' = \sup\{x;\ x \in \langle a, b\rangle,\ h(x) = 0\}$, $b' = \inf\{x;\ x \in \langle a, b \rangle,\ h(x) = 1\}$.

We are going to show that h is increasing on $\langle a', b'\rangle$. In the opposite case there are $c, d \in (a', b')$ such that $c < d$, $h(\langle c, d\rangle) = \{\alpha\}$. Clearly, $0 < \alpha < 1$. Defining $h_1(x) = h(x)$ and $h_1(x) = \alpha$ according as $a \leq x < d$ and $c < x \leq b$, respectively, we see that h_1 is a continuous on $\langle a, b\rangle$ and harmonic on $(a, d) \cup (c, b) = (a, b)$. The relations $h_1(a) = \alpha h(a)$, $h_1(b) = \alpha h(b)$, $h_1(c) = \alpha > \alpha^2 = \alpha h(c)$ contradict (III).

Suppose now that I is regular and fix an $x \in I$. The equalities

$$1 = \omega_x^I(\{a\}) + \omega_x^I(\{b\}) ,$$

$$h(x) = h(a)\,\omega_x^I(\{a\}) + h(b)\,\omega_x^I(\{b\}) =$$

$$= \omega_x^I(\{b\})$$

yield the implications (1) - (3).

1.3. <u>Corollary</u>. Let I be a regular interval. If there is a continuous increasing function h on $\overline{I}$ such that $\mathrm{Rest}_I\, h \in \mathcal{H}_I$, then $\omega_x^I = \partial I$ for every $x \in I$.

<u>Proof</u>. Suppose that there is such a function on $\overline{I} = \langle a, b\rangle$. Multiplying it by a suitable positive factor and subtracting a suitable constant one may clearly achieve that $h(a) = 0$, $h(b) = 1$, so that 1.2 is applicable. Since $a' = a$ and $b' = b$ now, the implications (1) - (3) show that $\mathrm{spt}\, \omega_x^I = \{a, b\}$ for any $x \in I$.

1.4. Corollary. Let h, a, b, a', b' have the meaning described in 1.2. Then $(a', b') \subset E$.

Proof. If $x \in (a', b')$ and J is an arbitrary regular interval with $x \in J \subset \bar{J} \subset (a', b')$, then, by 1.3,

$$\operatorname{spt} \omega_x^J = \partial J .$$

1.5. Proposition. E is open and dense in X .

Proof. Let x be an arbitrary point in E and fix a regular interval $I = (a, b)$ with $\operatorname{spt} \omega_x^I = \partial I$. Further choose an $h \in \mathcal{C}(\langle a, b \rangle)$ with $\operatorname{Rest}_I h \in \mathcal{H}_I$, $h(a) = 0$, $h(b) = 1$. Applying 1.2 and 1.4 we conclude that $x \in (a', b') \subset E$, so that E is open. According to 1.4, any regular interval contains points of E . Since regular intervals form a base of X , E is dense in X .

1.6. Lemma. Let I be an open interval and suppose that $g \in \mathcal{H}_I$ is not constant on I . Then any $h \in \mathcal{H}_I$ can be expressed in the form $h = \alpha g + \beta$, where $\alpha, \beta \in R^1$ are uniquely determined by h .

Proof. Let $h \in \mathcal{H}_I$. Choose $a < b$ in I so that $g(a) \neq g(b)$ and define $\alpha, \beta \in R^1$ by the equations

$$\alpha g(a) + \beta = h(a) , \tag{4}$$

$$\alpha g(b) + \beta = h(b) . \tag{5}$$

Then $h_1 = \alpha g + \beta \in \mathcal{H}_I$ and, by (III), $h_1 = h$ on (a, b). Let now x be an arbitrary point in I

and choose $a_1, b_1 \in I$ such that

$$a_1 < \min(x,a) < \max(x,b) < b_1 .$$

According to (IV),(III) we have again $g(a_1) \neq g(b_1)$ and defining $\alpha_1, \beta_1 \in R^1$ by the equations

$$\alpha_1 g(a_1) + \beta_1 = h_1(a_1) ,$$

$$\alpha_1 g(b_1) + \beta_1 = h_1(b_1) ,$$

we conclude as above that

$$y \in (a_1, b_1) \Longrightarrow \alpha_1 g(y) + \beta_1 = h_1(y) .$$

Letting $y = a$ and $y = b$ we obtain from (4),(5) that $\alpha_1 = \alpha$, $\beta_1 = \beta$, so that $\alpha g(x) + \beta = h(x)$.

1.7. Proposition. A bounded open interval $I \neq \emptyset$ is regular if and only if each $h \in \mathcal{H}_I$ is bounded on I.

Proof. Suppose first that $I = (a,b)$ is regular and choose an $h \in \mathcal{C}(\langle a,b\rangle)$ which is harmonic on I and satisfies the boundary conditions $h(a) = 0$, $h(b) = 1$. Then $g = \mathrm{Rest}_I\, h$ is non-constant on I, whence

$$\mathcal{H}_I = \{\alpha g + \beta ;\ \alpha, \beta \in R^1\} .$$

Since g is bounded, so are all elements of $\mathcal{H}_I$.

Suppose now that there is an unbounded $g_1 \in \mathcal{H}_I$. Then g_1 is non-constant and each function in

$$\mathcal{H}_I = \{\alpha g_1 + \beta ;\ \alpha, \beta \in R^1\}$$

is either constant or unbounded. Consequently, I is not regular.

1.8. Lemma. Let $I = (a,b)$ be a regular interval and suppose that $x \in I$, $\mathrm{spt}\, \omega_x^I = \{b\}$. If

$I^* = (a^*, b^*)$ is any regular interval such that

$$(6) \qquad a < a^* < x < b^* < b ,$$

then

$$spt\, \omega_x^{I^*} \neq \partial I^* \implies spt\, \omega_x^{I^*} = \{ b^* \} .$$

Proof. Suppose that there is a regular interval $I^* = (a^*, b^*)$ satisfying (6) such that $spt\, \omega_x^{I^*} = \{ a^* \}$. Choose an $h \in \mathcal{C}(\langle a, b \rangle)$ which is harmonic on (a, b) and satisfies the boundary conditions $h(a) = 0, h(b) = 1$. Further choose an $h^* \in \mathcal{C}(\langle a^*, b^* \rangle)$ which is harmonic on I^* and satisfies the boundary conditions $h^*(a^*) = 0$, $h^*(b^*) = 1$. In view of 1.2, $spt\, \omega_x^I = \{ b \}$ implies $h(\langle x, b \rangle) = \{1\}$. Similarly, $h^*(\langle a^*, x \rangle) = \{0\}$, because $spt\, \omega_x^{I^*} = \{a^*\}$ Defining $g(t) = 0$ and $g(t) = h^*(t)$ according as $a \leq t < x$ and $a^* < t \leq b^*$, we get a $g \in \mathcal{C}(\langle a, b^* \rangle)$ which is harmonic on $(a, x) \cup (a^*, b^*) = (a, b^*)$. The relations $g(a) = h(a) (= 0)$, $g(b^*) = h(b^*) (= 1)$ and $g(x) = 0 < 1 = h(x)$ contradict (III).

By symmetry, the following lemma is also valid.

1.9. Lemma. Let $I = (a, b)$ be a regular interval, $x \in I$, and suppose that $spt\, \omega_x^I = \{a\}$. Then for any regular interval $I^* = (a^*, b^*)$ satisfying (6) the following implication holds:

$$spt\, \omega_x^{I^*} \neq \partial I^* \implies spt\, \omega_x^{I^*} = \{ a^* \} .$$

1.10. Remark. Let us recall that a closed (relative to X) set $A \subset X$ is termed absorbent provided each point $x \in A$ has a neighborhood $\mathcal{U}_x$ such that $spt\ \omega_x^V \subset A$ for every regular set V with $x \in V \subset \overline{V} \subset \mathcal{U}_x$. We shall denote by F^+ the set of those $x \in X$ for which $X \cap \langle x, +\infty)$ is an absorbent set. Replacing $\langle x, +\infty)$ by $(-\infty, x\rangle$ we define analogously F^- .

1.11. Proposition. If $x \in X \setminus F^+$ and $b > x$, then there is an $a < x$ such that each $h \in \mathcal{H}_{(x,b)}$ possesses an extension $g \in \mathcal{H}_{(a,b)}$, $Rest_{(x,b)}\, g = h$.

Symmetrically, if $x \in X \setminus F^-$, then each function harmonic in some left-hand neighborhood of x can be harmonically continued across x to the right.

Proof. Fix $x \in X \setminus F^+$ and $b > x$. Then there is a regular interval $I = (a, b^*)$ such that

$$a < x < b^* < b,\ spt\ \omega_x^I \neq \{b^*\} .$$

Choose a $g^* \in \mathcal{C}(\langle a, b^*\rangle)$ which is harmonic on (a, b^*) and satisfies the boundary condition $g^*(a) = 0$, $g^*(b^*) = 1$. Then

$$g^*(x) = g^*(a)\, \omega_x^I(\{a\}) + g^*(b^*)\, \omega_x^I(\{b^*\}) =$$

$$= \omega_x^I(\{b^*\}) < 1 = g^*(b^*) ,$$

so that g^* is not constant on (x, b^*) . Given an arbitrary $h \in \mathcal{H}_{(x,b)}$ we have thus by 1.6

$$h = \alpha g^* + \beta \quad \text{on} \quad (x, b^*)$$

for suitable $\alpha, \beta \in R^1$. Defining $g(t) = \alpha g^*(t) + \beta$

and $g(t) = h(t)$ according as $a < t < b^*$ and $x < t < b$, we arrive at a $g \in \mathcal{H}_{(a,b)}$ with $Rest_{(x,b)}\, g = h$.

1.12. Corollary. Let $(c,d) \subset X$. Then $F^+ \subset \subset X \setminus (c,d)$ if and only if each $h \in \mathcal{H}_{(a,b)}$ with $(a,b) \subset (c,d)$ possesses an extension $g \in \mathcal{H}_{(c,b)}$. Symmetrically, in order that $F^- \cap (c,d) = \emptyset$, it is necessary and sufficient that each $h \in \mathcal{H}_{(a,b)}$ with $(a,b) \subset (c,d)$ possess an extension $g \in \mathcal{H}_{(a,d)}$.

Proof. Suppose that $F^+ \cap (c,d) = \emptyset$ and let $h \in \mathcal{H}_{(a,b)}$, $(a,b) \subset (c,d)$. We are going to prove that there is a $g \in \mathcal{H}_{(c,b)}$ with $Rest_{(a,b)}\, g = h$. Since this assertion is trivial when $a = c$ or h is constant (a,b), we shall assume that $c < a < b \leq d$ and h is not constant on (a,b). Let Λ denote the set of all $\lambda \in \langle c,a\rangle$ for which there exists an $h_\lambda \in \mathcal{H}_{(\lambda,b)}$ with $Rest_{(a,b)}\, h_\lambda = h$. If $\lambda_1 < \lambda_2$ are elements of Λ, then $Rest_{(\lambda_2,b)}\, h_{\lambda_1} = h_{\lambda_2}$ by 1.6. This implies that $\inf \Lambda \in \Lambda$ and 1.11 ensures $\inf \Lambda = c$.

Conversely, suppose that each $h \in \mathcal{H}_{(a,b)}$ with $(a,b) \subset (c,d)$ extends harmonically to the interval (c,b). We fix an arbitrary $x \in (c,d)$ and are going to prove that $x \notin F^+$. In the opposite case there would be a regular interval (a,b) with $x \in (a,b) \subset \langle a,b\rangle \subset (c,d)$ such that $spt\, \omega_x^{(a,b)} = \{b\}$. Choose an $h \in \mathcal{C}(\langle a,b\rangle)$ with $Rest_{(a,b)}\, h \in \mathcal{H}_{(a,b)}$, $h(a) = 0$, $h(b) = 1$, and let

$g \in \mathcal{H}_{(c,b)}$ coincide with h on (a, b). Then

$$g(\langle x, b\rangle) = h(\langle x, b\rangle) = \{1\}$$

by 1.2. Let now (a_1, b_1) be an arbitrary regular interval contained in (x, b). Since any harmonic function $g_1 \in \mathcal{H}_{(a_1, b_1)}$ extends harmonically to (c, b_1) and g is not constant on (c, b_1), we conclude by 1.6 that

$$g_1 = \alpha g + \beta = \alpha \quad \text{on} \quad (a_1, b_1)$$

for suitable $\alpha, \beta \in R^1$. We have thus shown that all functions in $\mathcal{H}_{(a_1, b_1)}$ are constant on (a_1, b_1), which contradicts the regularity of (a_1, b_1).

1.13. <u>Corollary.</u> If h is harmonic on an open interval J, then h is monotonous on J. If $J \subset E$ and $h \in \mathcal{H}_J$ is not constant on J, then h is 1-1 on J.

<u>Proof</u>. Suppose that $h \in \mathcal{H}_J$ and $h(x) < h(y)$ for a couple of points $x < y$ in $J = (c, d)$. It follows from 1.2 that h is non-decreasing on every interval (a, b) with $\{x, y\} \subset (a, b) \subset \langle a, b\rangle \subset J$ and, consequently, also on J. Suppose now that $J \subset E$ and consider an arbitrary regular interval $(a_1, b_1) \subset (c, d)$. It is sufficient to show that h is non-constant on (a_1, b_1).

In view of our assumptions, h is non-constant on one at least of the intervals (c, b_1), (a_1, d). For definiteness, suppose that h is non-constant on (c, b_1). By 1.12, any $g_1 \in \mathcal{H}_{(a_1, b_1)}$ extends harmonically to (c, b_1) and, in view of 1.6, $g_1 = \alpha h + \beta$ on

(a_1, b_1) for suitable $\alpha, \beta \in R^1$. Taking into account that (a_1, b_1) is regular we conclude from

$$\mathcal{H}_{(a_1,b_1)} = \{ \alpha \,\mathrm{Rest}_{(a_1,b_1)} h + \beta \,;\ \alpha, \beta \in R^1 \}$$

that h cannot be constant on (a_1, b_1).

1.14. Proposition. F^+, F^- are separated sets with $F^+ \cup F^- = X \setminus E$.

Proof. Clearly, $F^+ \cup F^- \subset X \setminus E$, $F^+ \cap F^- = \emptyset$. Consider now an arbitrary $x \in X \setminus E$ and fix a regular interval $(a, b) \subset X$ containing x such that

$$\mathrm{spt}\ \omega_x^{(a,b)} \neq \{a, b\} . \tag{7}$$

Let us distinguish the following two cases:

$$\mathrm{spt}\ \omega_x^{(a,b)} = \{b\} , \tag{8}$$

$$\mathrm{spt}\ \omega_x^{(a,b)} = \{a\} . \tag{9}$$

Choose an $h \in \mathcal{C}(\langle a, b\rangle)$ which is harmonic on (a, b) and satisfies the boundary conditions $h(a) = 0$, $h(b) = 1$. Further define a', b' as in Lemma 1.2. Consider first the case (8). Then $b' \leq x$ and we are going to prove that $\langle b', b) \subset F^+ \cup E$. Let $y \in \langle b', b\rangle$, so that $\mathrm{spt}\ \omega_y^{(a,b)} = \{b\}$ by 1.2. If $y \notin E$, then there is a neighborhood U_y of y such that

$$\mathrm{spt}\ \omega_y^{I^*} \neq \partial I^* \tag{10}$$

for each regular interval I^* satisfying

(11) $$y \in I^* \subset \overline{I^*} \subset \mathcal{U}_y \cap (a, b) \ .$$

Let V be an arbitrary regular set with $y \in V \subset \overline{V} \subset$ $\subset \mathcal{U}_y \cap (a, b)$ and denote by $I^* = (a^*, b^*)$ the component of V containing y, so that (11) holds. Employing (10), 1.8 and Remark 1.1 we get

$$\operatorname{spt} \omega_y^V = \operatorname{spt} \omega_y^{I^*} = \{b^*\} \subset \langle y, +\infty) \ ,$$

so that $y \in F^+$. Since $(a', b') \subset E$ by 1.4, we have thus verified the implication

$$(8) \Longrightarrow x \in (a', b) \subset E \cup F^+ \ .$$

Using 1.9 in place of 1.8 one concludes by a symmetric argument that

$$(9) \Longrightarrow x \in (a, b') \subset E \cup F^- \ .$$

Thus both F^+ and F^- are open in $F = X \setminus E$ and $F \subset F^+ \cup F^-$.

1.15. Lemma. $(X, \mathcal{H})$ satisfies the convergence axiom of J.L.Doob (see axiom III in [1],chap.I, § 1).

Proof. Consider an arbitrary regular set $\mathcal{U} \subset X$ and fix a component $I = (a, b)$ of $\mathcal{U}$. Writing ε_z for the unit point-mass (= Dirac measure) concentrated at z we have for any $x \in I$

$$\omega_x^{\mathcal{U}} = \omega_x^I(\{a\})\, \varepsilon_a + \omega_x^I(\{b\})\, \varepsilon_b$$

(see Remark 1.1 above). As shown in 1.2, there are $a' < b'$ in $\langle a, b \rangle$ such that the function

$$x \longrightarrow \omega_x^I(\{b\}) = h(x)$$

is increasing on (a', b') and $h(a'+) = 0$,

$h(b' -) = 1$.

Thus both $\omega_x^I(\{b\})$ and $\omega_x^I(\{a\}) = 1 - h(x)$ are positive for $x \in (a', b')$. If now f is a numerical (= extended real-valued) function on $\partial \mathcal{U}$ which is integrable $(\omega_x^{\mathcal{U}})$ for some $x \in (a', b')$, then f is necessarily finite on $\{a, b\} = \partial I$ and the function

$$x \longrightarrow \int f \, d\omega_x^{\mathcal{U}} = f(b)\, h(x) + f(a)\,[1 - h(x)]$$

is harmonic on I . In particular, if f is integrable $(\omega_x^{\mathcal{U}})$ for all x in some dense subset of $\mathcal{U}$, then f must be finite on $\partial \mathcal{U}$ (and, consequently, integrable $(\omega_x^{\mathcal{U}})$ for all $x \in \mathcal{U}$) and the function

$$x \longrightarrow \int f \, d\omega_x^{\mathcal{U}}$$

is harmonic on $\mathcal{U}$. Thus the equivalent form III'' of J. L. Doob 's convergence axiom (see Theorem 1.1.8 in [1]) has been verified.

1.16. <u>Remark</u>. It follows from (I),(II) and 1.15 that $(X, \mathcal{H})$ satisfies the first three axioms of H. Bauer 's axiomatic theory as formulated in [1], chap. I, § 1. The last axiom of this theory, however, is fulfilled only locally (see 1.19, 1.20 below).

If one adopted all axioms of H. Bauer, then in 1.14 more could be said about F^+, F^- , as shown by the following proposition.

1.17. <u>Proposition</u>. Suppose, in addition, that $(X, \mathcal{H})$ satisfies the axiom IV of H. Bauer's axiomatic theory [1]. Then either $F^- = \emptyset$ or $F^+ = \emptyset$ or else $F^- \neq \emptyset \neq F^+$ and $\sup F^- < \inf F^+$.

A closed set (relative to X) $A \subset X$ is absorbent if and only if $X \setminus A = (\alpha, \beta)$, where either $\alpha \notin X$ or else $\alpha \in F^-$ and, similarly, either $\beta \notin X$ or else $\beta \in F^+$.

Proof. The first part of this assertion follows from the following reasoning which was communicated to us by C. Constantinescu and A. Cornea. If $x \in F^-$ and $y \in F^+$, then $x \leq y$, because in the opposite case $\langle y, x \rangle = X \cap \langle y, +\infty) \cap (-\infty, x\rangle \neq \emptyset$ would be a compact absorbent set, which cannot occur in the Bauer theory. Since F^-, F^+ are closed in X by 1.14 and 1.5, we conclude that $\sup F^- < \inf F^+$ provided $F^- \neq \emptyset \neq F^+$.

Let now $A \subsetneqq X$ be an absorbent set and put $B = X \setminus A$. Consider an arbitrary couple of points $x < y$ in B. It follows easily from the definition of an absorbent set that

$$\langle x, y \rangle \cap A = (x, y) \cap A$$

is again an absorbent set. Since $\langle x, y \rangle \cap A$ is compact, we conclude that $\langle x, y \rangle \subset B$, so that B is an interval. The rest follows easily from the definition of F^- and F^+.

1.18. Remark. A numerical function u is termed hyperharmonic on an open set $\mathcal{U} \subset X$ provided u is lower-semicontinuous and $> -\infty$ on $\mathcal{U}$ and each $x \in \mathcal{U}$ possesses a neighborhood $\mathcal{U}_x \subset X$ such that

$$u(x) \geq \int u \, d\omega_x^V$$

whenever V is a regular set with $x \in V \subset \overline{V} \subset U_x$.

The class of all hyperharmonic functions on U is denoted by $\mathcal{H}_U^*$ and ${}_+\mathcal{H}_U^*$ is used to denote the subclass of all non-negative functions in $\mathcal{H}_U^*$.

1.19. Lemma. Every $x \in X$ is contained in an open interval $J \subset X$ such that ${}_+\mathcal{H}_J^*$ separates the points of J .

Proof. Fix $x \in X$ and choose an open interval $J = (a, b)$ containing x with $\{a, b\} \subset E$ such that one at least of the sets $F^- \cap J$, $F^+ \cap J$ is empty (see 1.14 and 1.5). For definiteness, suppose that $F^- \cap J = \emptyset$. Consider an arbitrary couple of points $y < z$ in J. Let $J_1 = (a_1, b_1)$ be a regular interval, $y < a_1 < b_1 < z$, and choose an $h \in \mathcal{H}_{J_1}$ with $h(a_1+) = 1$, $h(b_1-) = 0$. Put $c = a_1$ if $a_1 \in F^+$, and in the opposite case let

$$c = \inf\{t \,;\, t \in (a, a_1\rangle,\ F^+ \cap (t, a_1\rangle = \emptyset\} .$$

Further choose a $d > b$ such that $\langle b, d\rangle \subset E$. According to 1.12, h extends to a harmonic function g on (c, d). By 1.13, g is non-increasing on (c, d) and, consequently, bounded from below on (c, b). Choose a $k \in R^1$ such that $g_1 = k + g$ is positive on (c, b). If $c > a$, then $c \in F^+$ and we extend g_1 to (a, b) defining $g_1(t) = +\infty$ for $a < t \leq c$. Thus we obtain a $g_1 \in {}_+\mathcal{H}_J^*$ with

$$g_1(y) \geq g_1(a_1) > g_1(b_1) \geq g_1(z) .$$

1.20. Corollary. Every $x \in X$ is contained in an open interval $J \subset X$ such that $(J, \mathrm{Rest}_J \mathcal{H})$ (where $\mathrm{Rest}_J \mathcal{H}$ denotes the restriction of $\mathcal{H}$ to the system of open sets contained in J) is a harmonic space in the sense of H. Bauer [1].

Proof. This follows from 1.19 and 1.16.

1.21. Lemma. Suppose that $\mathcal{H}$ satisfies the axioms (II), (III) stated above and, instead of (III) and (IV), assume that the following axiom (III*) is fulfilled:

(III*) Every $x \in X$ is contained in an open interval $\mathcal{U}_x \subset X$ such that there is a (strictly) positive harmonic function on $\mathcal{U}_x$ and, for each compact interval $J \subset \mathcal{U}_x$, the following minimum principle is satisfied:

If $h \in \mathcal{C}(J)$ is harmonic on the interior of J and non-negative on ∂J, then $h \geq 0$ on J.

If, moreover, $X = E$, then any $h \in \mathcal{H}_X$ vanishes identically on X provided $\{x ; x \in X, h(x) = 0\}$ has an accumulation point in X and, for any interval $I \subset X$, each $g \in \mathcal{H}_I$ extends to a uniquely determined $\tilde{g} \in \mathcal{H}_X$.

Proof. It is easily seen that there is a sequence $x_n \in X$ such that the corresponding intervals $\mathcal{U}_{x_n}$ (occurring in (III*)) form a covering of X and $\bigcup_{n=1}^{k} \mathcal{U}_{x_n}$ is an interval for each positive integer k. Given an open interval $I \neq \emptyset$, we may clearly assume that $I \cap \mathcal{U}_{x_1} \neq \emptyset$. Fix an n and consider

$\mathcal{U}_{x_n} = \mathcal{U}$. Since there is a harmonic function $h > 0$ on $\mathcal{U}$ we may introduce the so-called h-harmonic functions on $\mathcal{U}$ passing by the standard procedure from $(\mathcal{U}, \mathrm{Rest}_{\mathcal{U}}\, \mathcal{H})$ to $(\mathcal{U}, {}^{h}\widetilde{\mathcal{H}})$, where

$$ {}^{h}\widetilde{\mathcal{H}}_G = \{ g/h \; ; \; g \in \mathcal{H}_G \} $$

for each open $G \subset \mathcal{U}$. Then, as it is well known, regular sets in $(\mathcal{U}, {}^{h}\widetilde{\mathcal{H}})$ are just the same as those regular in $(\mathcal{U}, \mathrm{Rest}_{\mathcal{U}}\, \mathcal{H})$ and, for any regular V , the corresponding harmonic measures $\widetilde{\omega}_x^V$ and ω_x^V in $(\mathcal{U}, {}^{h}\widetilde{\mathcal{H}})$ and $(\mathcal{U}, \mathrm{Rest}_{\mathcal{U}}\, \mathcal{H})$ respectively, satisfy

$$ \widetilde{\omega}_x^V = \frac{1}{h(x)} (h\, \omega_x^V) \; , \quad x \in V \; . $$

Hence it follows that the set of all elliptic points of $(\mathcal{U}, {}^{h}\widetilde{\mathcal{H}})$ coincides with $\mathrm{E} \cap \mathcal{U} = \mathcal{U}$. Applying 1.13 and 1.6 to the harmonic space $(\mathcal{U}, {}^{h}\widetilde{\mathcal{H}})$ we obtain that there is an increasing h-harmonic function φ on $\mathcal{U}$ such that

$$ {}^{h}\widetilde{\mathcal{H}}_{\mathcal{U}} = \{ \alpha \varphi + \beta ; \; \alpha, \beta \in \mathrm{R}^1 \} \; , $$

whence

$$ \mathcal{H}_{\mathcal{U}} = \{ \alpha \varphi h + \beta h ; \; \alpha, \beta \in \mathrm{R}^1 \} \; . $$

Consequently, every $q \in \mathcal{H}_{\mathcal{U}}$ vanishes identically on $\mathcal{U}$ provided q has more than one zero in $\mathcal{U}$. It is also easily seen that any harmonic function defined on an interval contained in $\mathcal{U}$ extends harmonically to the whole of $\mathcal{U}$ (cf. 1.12).

To complete the proof we start with an arbitrary $g \in \mathcal{H}_I$ and extend it harmonically from $I \cap \mathcal{U}_{x_1}$ to

$\mathcal{U}_{x_1}$ and then, consecutively, to $\bigcup_{n=1}^{k} \mathcal{U}_{x_n}$ for any $k = 2, 3, \dots$. Finally we arrive at a $\tilde{g} \in \mathcal{H}_X$. It is easily seen that g is uniquely determined and $\tilde{g} = g$ on I .

§ 2.

Throughout this paragraph X will denote a one-dimensional manifold, i.e. a Hausdorff topological space each point of which has a neighborhood homeomorphic with the real line R^1 . By a compact arc we mean a set homeomorphic with the interval $\langle 0, 1 \rangle \subset R^1$, by an (open) arc we mean a set homeomorphic with R^1 . We shall suppose that with each open set $\mathcal{U} \subset X$ there is associated a vector space $\mathcal{H}_{\mathcal{U}}$ (over the real number field) of continuous real-valued functions (called harmonic functions) such that the map $\mathcal{H} : \mathcal{U} \longrightarrow \mathcal{H}_{\mathcal{U}}$ satisfies the sheaf axiom and the basis axiom (see (I) and (II) in § 1) as well as the following axiom:

III* . Each $x \in X$ possesses a neighborhood $\mathcal{U}_x$ such that there is a (strictly) positive harmonic function on $\mathcal{U}_x$ and, for each compact arc $C \subset \mathcal{U}_x$, the following form of the minimum principle is fulfilled:

If $h \in \mathcal{C}(C)$ is non-negative on ∂C and harmonic on the interior of C , then $h \geq 0$ on C .

2.1. <u>Remark</u>. The above requirements are met if $(X, \mathcal{H})$ is a harmonic space in the sense of N. Boboc, C. Constantinescu and A. Cornea [2] (see axioms H_0 - H_3 on p. 283 and corollary 1.2 on p. 287).

Conversely, for the special $\mathcal{X}$ considered here, the axioms (I), (II), III* imply that $(\mathcal{X}, \mathcal{H})$ fulfils locally the axioms of H. Bauer and, consequently, $(\mathcal{X}, \mathcal{H})$ satisfies the axioms $H_0 - H_3$ of N. Boboc, C. Constantinescu and A. Cornea. Indeed, using 1.20 one concludes easily that each $x \in \mathcal{X}$ is contained in an arc C such that $(C, Rest_C\, \mathcal{H})$ is a harmonic space in the sense of H. Bauer.

Proposition 1.5 permits us to prove the following theorem announced in the introduction.

2.2. Theorem. Let $E(\mathcal{X}, \mathcal{H})$ denote the set of all elliptic points of $(\mathcal{X}, \mathcal{H})$. Then $E(\mathcal{X}, \mathcal{H})$ is an open dense subset of $\mathcal{X}$ and each component of $E(\mathcal{X}, \mathcal{H})$ has a countable base.

Proof. In order to prove that $E(\mathcal{X}, \mathcal{H})$ is open and dense in $\mathcal{X}$ we shall show that each $x \in \mathcal{X}$ is contained in an arc $\mathcal{U}$ such that $\mathcal{U} \cap E(\mathcal{X}, \mathcal{H})$ is open and dense in $\mathcal{U}$. Let $x \in \mathcal{X}$ and choose an arc $\mathcal{U}_x = \mathcal{U} \ni x$ possessing the properties described in III* . Since there is a harmonic function $h > 0$ on $\mathcal{U}$, we may introduce the harmonic space $(\mathcal{U}, {}^h\widetilde{\mathcal{H}})$ formed by h-harmonic functions concluding that

$$E(\mathcal{U}, {}^h\widetilde{\mathcal{H}}) = \mathcal{U} \cap E(\mathcal{X}, \mathcal{H}) .$$

Since constants are h-harmonic on $\mathcal{U}$ and $\mathcal{U}$ is homeomorphic with an open interval in R^1 we obtain from 1.5 that $E(\mathcal{U}, {}^h\widetilde{\mathcal{H}})$ is open and dense in $\mathcal{U}$.

It remains to verify that each component of $E(\mathcal{X}, \mathcal{H})$ has a countable base. Since this is clear if $\mathcal{X}$ is compact, we shall now assume that $\mathcal{X}$ is non-compact and connected and $X = E(X, \mathcal{H})$.

Let us first notice that any harmonic function g defined on an arc $U \subset X$ extends to a uniquely determined harmonic function on X . This follows easily from Lemma 1.21 which guarantees that for each arc $C \supset \supset U$ there is a uniquely determined harmonic extension of g to C .

Using this observation we fix an arc U and a $g \in \mathcal{H}_X$ such that $g > 0$ on U . Put

$$H = \{x ;\ x \in X,\ g(x) = 0\} ,$$

so that H is closed and, according to 1.21, has no accumulation points in X . We are going to prove that H is at most countable. Fix an $x_0 \in U$, denote by D any of the two components of $X \setminus \{x_0\}$ and suppose that $H \cap D$ is infinite. For any $y \in D$ there is precisely one compact arc C_y with $\partial C_y = \{x_0, y\}$ and we may define a linear order on D by

$$y_1 \preceq y_2 \iff C_{y_1} \subset C_{y_2} .$$

Using the fact that $H \cap C_y$ is finite for each $y \in D$ one easily concludes that there is a similarity of the ordered set $(H \cap D, \preceq)$ onto the set of all positive integers.

We see that $X \setminus H$ has at most countably many components and it is sufficient to show that each of these components has a countable base. Let G be an arbitrary component of $X \setminus H$. Then there is a positive harmonic function h $(= \pm g)$ on G and we may consider h-harmonic functions on G . Since constants are h-harmonic and any h-harmonic function defined on an arc contained in G extends (uniquely) to an h-harmonic function on G, we conclude that there is an h-harmonic function h_G on G which is locally 1-1 on G (see 1.13). Then h_G maps G homeomorphically onto $h_G(G) \subset \mathbb{R}^1$ and, consequently, G has a countable base.

2.3. Corollary. If X is a connected 1-manifold and $(X, \mathcal{H})$ satisfies the axioms of the Brelot theory of harmonic spaces ([3], [4], [5]), then X necessarily has a countable base.

2.4. Notation. Making use of the orientability of each component of X we suppose that for each arc $C \subset X$ there has been fixed a distinguished homeomorphism $\varphi_C : C \longrightarrow \varphi_C(C) \subset \mathbb{R}^1$ such that $\varphi_{C_2} \circ \varphi_{C_1}^{-1}$ (= the composite of $\varphi_{C_1}^{-1}$ and φ_{C_2}) is an increasing function on each component of $\varphi_{C_1}(C_1 \cap C_2)$ whenever the arcs $C_1, C_2 \subset X$ have non-void intersection.

If C is an arc and $x \in C$, then $C^-(x)$ and $C^+(x)$ will denote the components of $C \setminus \{x\}$ with the notation so chosen that $\varphi_C(C^-(x))$ and $\varphi_C(C^+(x))$ are, respectively, the left-hand and the right-hand components of $\varphi_C(C) \setminus \{\varphi_C(x)\}$.

We shall denote by $F^-(X, \mathcal{H})$ the set of those $x \in X$ for which there is an arc $C \ni x$ such that $\operatorname{spt} \omega_x^V \subset C^-(x)$ whenever V is regular and $x \in V \subset \overline{V} \subset C$. The set $F^+(X, \mathcal{H})$ is defined similarly (cf. the introduction).

Employing Proposition 1.14 one easily obtains the following result.

2.5. <u>Theorem</u>. $F^-(X,\mathcal{H})$, $F^+(X,\mathcal{H})$ are disjoint closed subsets of X with $F^-(X,\mathcal{H}) \cup$ $\cup F^+(X,\mathcal{H}) = X \setminus E(X,\mathcal{H})$.

We are now going to show that the sets $E(X,\mathcal{H})$, $F^-(X,\mathcal{H})$ and $F^+(X,\mathcal{H})$ are completely characterized by the properties described in Theorems 2.2, 2.5.

2.6. <u>Theorem</u>. Given disjoint closed sets $F^-, F^+ \subset$ $\subset X$ such that $E = X \setminus (F^- \cup F^+)$ is dense in X and each component of E has a countable base, then there is a harmonic sheaf $\mathcal{H}$ on X such that $F^- = F^-(X,\mathcal{H})$, $F^+ = F^+(X,\mathcal{H})$.

<u>Proof</u>. We may clearly suppose that X is connected. Suppose first that X is non-compact. If $x, y \in X$, then there is always an arc $C \supset \{x, y\}$ and we shall define

$$x \preceq y \quad \text{if and only if} \quad \varphi_C(x) \leq \varphi_C(y) .$$

Since now any two arcs in X intersect in a connected set one easily concludes from the properties of φ_C that this definition makes sense and $\preceq$ is a linear order on X such that the intervals $\{x ;\ y \prec x \prec z\}$ x) form a base of the topological space X .

The system $\mathcal{C}$ of all components of $E = X \setminus (F^- \cup F^+)$ splits into four subsystems $\mathcal{C}(+,-)$, $\mathcal{C}(-,+)$, $\mathcal{C}(+)$ and $\mathcal{C}(-)$ defined as follows:

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x) Here, of course, $y \prec x$ means that $y \preceq x$ and $y \neq x$.

$\mathscr{C}(+,-)$ is the system of all $C \in \mathscr{C}$ with $\inf C \in F^+$ and $\sup C \in F^-$, $\mathscr{C}(+)$ is the system of all $C \in \mathscr{C}$ with $\partial C \subset F^+$, $\mathscr{C}(-,+)$ and $\mathscr{C}(-)$ being defined in the same way with the role of F^+ and F^- interchanged. Note that each $C \in \mathscr{C}$ is an arc because it has a countable base. We now associate with each $C \in \mathscr{C}$ a function h_C defined on X in the following manner.

If $C \in \mathscr{C}(-)$, we fix a continuous (strictly) increasing real-valued function h_C on C with $\inf h_C(C) = 0$, $\sup h_C(C) = +\infty$ and extend it to X letting $h_C(X \setminus C) = \{0\}$.

If $C \in \mathscr{C}(+)$, then h_C is a continuous decreasing function on C, $\inf h_C(C) = 0$, $\sup h_C(C) = +\infty$, $h_C(X \setminus C) = \{0\}$. (Thus h_C need not, in general, be continuous at points of ∂C.)

If $C \in \mathscr{C}(+,-)$, then h_C is a continuous and decreasing function on C, $\inf h_C(C) = -\infty$, $\sup h_C(C) = +\infty$, $h_C(X \setminus C) = \{0\}$.

Finally, if $C \in \mathscr{C}(-,+)$ and $\inf C = x\ (\in F^-)$, $\sup C = y\ (\in F^+)$, then we fix a continuous increasing function h_C on $\overline{C}$ and extend it to X defining $h_C(z) = h_C(x)$ or $h_C(z) = h_C(y)$ according as $z \preceq x$ or $y \preceq z$, respectively.

We shall say that a function h is harmonic on an (open) arc a if it is continuous and expressible in the form

$$(12) \qquad h = k + \sum_{C \in \mathcal{C}} k_C h_C \quad \text{on } Q ,$$

where k, k_C $(C \in \mathcal{C})$ are real constants, $k_C = 0$ for all but a finite number of C's in $\mathcal{C}$. If $\mathcal{U} \subset X$ is open then the class $\mathcal{H}_{\mathcal{U}}$ (of all harmonic functions on $\mathcal{U}$) will consist of those functions that are harmonic on every arc $Q \subset \mathcal{U}$.

Consider now any open relatively compact arc Q which is small enough in the sense that one at least of the sets $F^- \cap \overline{Q}$ and $F^+ \cap \overline{Q}$ is empty. Suppose, for instance, that $F^+ \cap \overline{Q} = \emptyset$. Let $h \in \mathcal{H}_Q$.

The constants k_C $(C \in \mathcal{C})$ occurring in the representation (12) satisfy the implication

$$\sup C \in Q \Longrightarrow k_C = 0$$

and h_C is constant on Q whenever $\sup C \trianglelefteq \inf Q$ or $\sup Q \trianglelefteq \inf C$. We see that constants are the only functions harmonic on the whole of Q if there is no subarc $Q_1 \subset Q$ with $Q_1 \cap F^- = \emptyset$ and $\sup Q_1 = \sup Q$. Suppose now that there is such an arc $Q_1 \subset Q$, $\sup Q_1 = \sup Q$, $Q_1 \cap F^- = \emptyset$, and denote by C_Q that component of E which contains Q_1 . Then every $h \in \mathcal{H}_Q$ has the form

$$(13) \qquad h = k + k_{C_Q} h_{C_Q} \quad \text{on } Q$$

for suitable constants k, k_{C_Q} and one concludes easily that Q is regular if and only if $\sup Q \in C_Q$ or, which is the same,

$$(14) \qquad \sup Q \notin F^- .$$

Assuming (14) and noting that h_{C_Q} remains constant on $Q \setminus C_Q$ and increases on $Q \cap C_Q$ we arrive at the implications

$$x \in F^- \cap Q \Longrightarrow x \trianglelefteq \inf C_Q \Longrightarrow \operatorname{spt} \omega_x^Q = \{\inf Q\} .$$

We see that

$$(15) \qquad F^- \subset F^-(X, \mathcal{H}) .$$

A similar reasoning yields the inclusion

$$(16) \qquad F^+ \subset F^+(X, \mathcal{H}) .$$

Suppose now that $x \in E$ and let Q be an open relatively compact arc such that $x \in Q \subset \overline{Q} \subset E$. Using the notation introduced above we have then $\overline{Q} \subset C_Q$, so that h_{C_Q} is increasing on $\overline{Q}$. Consequently, Q is regular and

$$\operatorname{spt} \omega_x^Q = \partial Q .$$

We have thus

$$(17) \qquad E \subset E(X, \mathcal{H})$$

which together with (15),(16) implies $F^- = F^-(X, \mathcal{H})$, $F^+ = F^+(X, \mathcal{H})$. We leave it to the reader to verify in detail that $(X, \mathcal{H})$ satisfies locally the axioms of H. Bauer [1].

Let us remark that in the case when $F = F^- \cup F^+$ is compact the above construction can be modified so as to yield a harmonic sheaf $\overline{\mathcal{H}}$ on the Aleksandrov compactification $\overline{X} = X \cup \{\infty\}$ of X such that

$$E(\overline{X}, \overline{\mathcal{H}}) = E \cup \{\infty\}, F^+(\overline{X}, \overline{\mathcal{H}}) = F^+, F^-(\overline{X}, \overline{\mathcal{H}}) = F^- .$$

Indeed, suppose that F is compact and non-void (investigation of the case $F = \emptyset$ is simple and may be left to the reader) and put $x_1 = \inf F$, $x_2 = \sup F$,

$$C_1 = \{x; x \in X, x \prec x_1\}, \quad C_2 = \{x; x \in X, x_2 \prec x\} .$$

For $\mathcal{U} \subset X$ we leave the definition of $\overline{\mathcal{H}}_{\mathcal{U}} = \mathcal{H}_{\mathcal{U}}$ unchanged, as well as the construction of h_C for $C \in \mathcal{C} \setminus \{C_1, C_2\}$, while the definition of h_{C_1}, h_{C_2} will be modified slightly. It is sufficient to define harmonic functions on subdomains of $C_1 \cup \{\infty\} \cup C_2$ (in $\overline{X}$) containing ∞ and we shall agree that these will be the functions which arise as restrictions of functions harmonic on $D = C_1 \cup \{\infty\} \cup C_2$ forming the vector space $\overline{\mathcal{H}}_D$ to be defined below. Let us distinguish the following cases a) - d):

a) If $x_1 \in F^+$ and $x_2 \in F^+$, then $C_1 \in \mathcal{C}(+)$ and making no change in the definition of h_{C_2} we now require that h_{C_1} be bounded, continuous and decreasing on C_1,

$$0 = \inf h_{C_1}(C_1), \quad \sup h_{C_1}(C_1) = 1 .$$

By definition, $h \in \mathcal{H}_D$ if and only if

$$\mathrm{Rest}_{C_1} h = k_0 + k_1 h_{C_1} \quad \text{on } C_1 ,$$

$$\mathrm{Rest}_{C_2} h = k_0 + k_1 (1 + h_{C_2}) \text{ on } C_2 ,$$

$$h(\infty) = k_0 + k_1 ,$$

where $k_0, k_1 \in R^1$.

b) If $x_1 \in F^+$ and $x_2 \in F^+$, then we choose h_{C_1} as in a) and require that h_{C_2} be bounded, continuous and increasing on C_2,

$$0 = \inf h_{C_2}(C_2), \quad \sup h_{C_2}(C_2) = 1 .$$

Now h will be termed harmonic on D if

$$\mathrm{Rest}_{C_1} h = k_0 + k_1 h_{C_1} \quad \text{on } C_1 ,$$

$$\mathrm{Rest}_{C_2} h = 2k_1 + k_0 - k_1 h_{C_2} \text{ on } C_2 ,$$
$$h(\infty) = k_0 + k_1 ,$$

where $k_0, k_1 \in R^1$.

c) If $x_1 \in F^-$ and $x_2 \in F^-$, then $C_1 \in \mathscr{C}(-)$ and no change is made in the original definition of h_{C_1}, while h_{C_2} is chosen in the same way as in the case b). Now $\overline{\mathscr{H}}_D$ will consist of those h for which

$$\mathrm{Rest}_{C_2} h = k_1 + k_2 h_{C_2} \quad \text{on } C_2 ,$$

$$\mathrm{Rest}_{C_1} h = k_1 + k_2 (1 + h_{C_1}) \text{ on } C_1 ,$$

$$h(\infty) = k_1 + k_2 ,$$

where $k_1, k_2 \in R^1$.

d) If $x_1 \in F^-$ and $x_2 \in F^+$, then $C_1 \in \mathscr{C}(-)$ and $C_2 \in \mathscr{C}(+)$. Retaining the original definition of h_{C_1} and h_{C_2} we term h harmonic on D provided

$$\mathrm{Rest}_{C_1} h = k_0 + k_1 h_{C_1} \quad \text{on } C_1 ,$$

$$\mathrm{Rest}_{C_2} h = k_0 - k_1 h_{C_2} \quad \text{on } C_2 ,$$

$$h(\infty) = k_0 ,$$

where $k_0, k_1 \in \mathbb{R}^1$.

We leave it to the reader to verify that this construction turns $X = X \cup \{\infty\}$ into a harmonic space satisfying locally the axioms of H. Bauer [1] with $E(\overline{X}, \overline{\mathcal{H}}) = E \cup \{\infty\}$.

It remains to consider the case when X is compact. We may then fix an $x_0 \in E$, so that $F = F^+ \cup$ $\cup F^- \subset X_0 = X \setminus \{x_0\}$. Now F is compact in the non-compact space X_0 and the above remark may be employed to get the desired harmonic sheaf on the Aleksandrov compactification $X_0 \cup \{x_0\} = X$.

2.7. <u>Remark</u>. In general, the set of all elliptic points need not be open. This is shown by the following example which was communicated to us by C. Constantinescu and A. Cornea.

Let $X = \mathbb{R}^1, F^- = \emptyset, F^+ = \{0\} \cup \{\frac{1}{n} ; n = 1, 2, \dots\}$.

According to the above theorem, there is a harmonic sheaf $\mathcal{H}$ on $\mathbb{R}^1$ such that $E(X, \mathcal{H}) = X \setminus F^+$, $F^+(X, \mathcal{H}) = F^+$. Since $K = \langle 0, +\infty)$ is an absorbent set, one may consider it as a new harmonic space whose harmonic functions arise as restrictions to K of functions harmonic in X (see C. Constantinescu [6]). Then 0 is an elliptic point of K and the points $\frac{1}{n}$ $(n = 1, 2, \dots)$ tending to 0 are not elliptic.

2.8. <u>Theorem</u>. A closed set $A \subset X$ is absorbent if and only if each $x \in \partial A$ is contained in an arc

$C = C_x \subset X$ such that either $A \supset C^+(x)$ and $x \in F^+(X, \mathcal{H})$ or else $A \supset C^-(x)$ and $x \in \in F^-(X, \mathcal{H})$.

Proof. This follows easily from 1.17 and the fact that $(X, \mathcal{H})$ satisfies locally the axioms of H. Bauer (see 2.1).

R e f e r e n c e s

[1] H. BAUER: Harmonische Räume and ihre Potentialtheorie, Springer Verlag, 1966.

[2] N. BOBOC, C. CONSTANTINESCU and A. CORNEA: Axiomatic theory of harmonic functions. Non-negative superharmonic functions, Annales de l'Institut Fourier XV(1965),283-312.

[3] M. BRELOT: Lectures on potential theory, Tata Institute of Fundamental Research, Bombay,1960.

[4] M. BRELOT: Axiomatique des fonctions harmoniques, Université de Montréal, 1966.

[5] C. CONSTANTINESCU and A. CORNEA: On the axiomatic of harmonic functions, Annales de l'Institut Fourier XIII(1963),373-388.

[6] C. CONSTANTINESCU: Harmonic spaces. Absorbent sets and balayage, Rev.Roum.Math.Pures et Appl. 11(1966),887-910.

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