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SOME MAPPING THEOREMS AND SOLVABILITY OF NONLINEAR EQUATIONS IN BANACH SPACE

(Preliminary communication)

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In this note we are concerned with some mapping theorems and solvability of nonlinear operator equations in Banach spaces. For the recent results in these topics see for instance Browder [1], Brezis [2], Petryshyn [3] and others.

Let X, Y be normed linear spaces. We use " $\longrightarrow$ ", " $\xrightarrow{w}$ " to denote strong and weak convergence, respectively. A mapping $F: X \longrightarrow Y$ is said to be

weakly continuous if

$u_n, u \in X, u_n \xrightarrow{w} u \Longrightarrow F(u_n) \xrightarrow{w} F(u)$;

demicontinuous if

$u_n, u \in X, u_n \longrightarrow u \Longrightarrow F(u_n) \xrightarrow{w} F(u)$;

p-positively homogeneous if $F(tu) = t^p F(u)$ for each $u \in X$, $t \geq 0$, where $p > 0$.

We shall say that a functional φ is quasi-convex on a convex set $M \subset X$, if $u, v \in M$, $\lambda \in [0,1] \Longrightarrow$ $\Longrightarrow \varphi(\lambda u + (1-\lambda) v) \leq \max(\varphi(u), \varphi(v))$. By $B_\sigma(u)$,

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$\partial B_\sigma(u)$ we denote an open ball centered about u and with the radius $\sigma > 0$, its boundary, respectively.

<u>Theorem 1</u>. Let X be a reflexive Banach space, $F: X \rightarrow X$ a weakly continuous mapping so that $F(0) = 0$. Assume $E(a) = \{u \in X : \|F(u)\| \leq a\}$ is bounded for each $a > 0$. Let for some $\lambda > 0$, $u, v \in X$,

$$u \neq v \Longrightarrow \|u - v - \lambda(F(u) - F(v))\| < \|u - v\| .$$

Then $F(X) = X$; i.e. for each $y \in X$ there exists at least one $u \in X$ so that $F(u) = y$.

<u>Remark</u>. The condition that $E(a)$ is bounded for each $a > 0$ is satisfied for instance when F is p-positively homogeneous operator on X and $\|F(u)\| \geq$ $\geq m > 0$ for each $u \in X$, $\|u\| = r$ (r positive and fixed).

<u>Theorem 2</u>. Let X, Y be normed linear spaces, X reflexive, $F: X \rightarrow Y$ a weakly continuous mapping, $F(0) = 0$ and so that $E(a) = \{u \in X : \|F(u)\| \leq a\}$ is bounded for each $a > 0$. Let $H: X \rightarrow Y$ be a p-positively homogeneous mapping of X onto Y. Suppose that for each point $u \in X$ there exist constants α_u, δ_u $(0 \leq \alpha_u < 1,\ 0 < \delta_u)$ and a mapping $G_u : X \rightarrow Y$ so that $v \in B_{\delta_u}(u) \Longrightarrow \|F(v) - F(u) -$ $- G_u(v - u)\| \leq \alpha_u \|H(v - u)\|$. Assume there exists $R > 0$ and $\varepsilon_u > 0$ so that $v \in B_R(0) \Longrightarrow$ $\Longrightarrow \|G_u(v) - H(v)\| \leq \varepsilon_u \|Hv\|$ for each $u \in X$. If $\varepsilon_u + \alpha_u < 1$ for each $u \in X$, then $F(X) = Y$.

<u>Corollary 1</u>. Let X, Y be normed linear spaces,

$F: X \to Y$ a weakly continuous mapping so that $E(a) = \{u \in X : \|F(u)\| \leq a\}$ is bounded for each $a > 0$ and $F(0) = 0$. Let G be p-positively homogeneous mapping of X onto Y. Assume that for each point $u \in X$ there exist the constants α_u, δ_u $(0 \leq \alpha_u < 1, \delta_u > 0)$ so that $v \in B_{\delta_u}(u) \Longrightarrow \|F(v) - F(u) - G(v-u)\| \leq \alpha_u \|G(v-u)\|$. Then $F(X) = Y$.

<u>Theorem 3</u>. Let X, Y be normed linear spaces, X reflexive, $M \subset X$ open, $F: M \to Y$ a weakly continuous map on M, $G: X \to Y$ p-positively homogeneous mapping onto Y so that for each $u_1, u_2 \in M \Longrightarrow \|F(u_1) - F(u_2) - G(u_1 - u_2)\| \leq \alpha \|G(u_1 - u_2)\|$, where $0 \leq \alpha < 1$. Suppose that for each point $u_0 \in M$ there exist the numbers $\delta_{u_0} > 0$, $a_{u_0} > 0$ so that $\overline{B_{\delta_{u_0}}(u_0)} \subset M$ and $u \in \partial B_{\delta_{u_0}}(u_0) \Longrightarrow$ $\Longrightarrow \|F(u_0) - F(u)\| \geq a_{u_0} \|u - u_0\| = a_{u_0} \delta_{u_0}$. Then $F(M)$ is open.

Let X, Y be normed linear spaces. Following [4] a mapping $K: X \to Y$ is said to be relatively open if for each open $E \subset X$ the set $K(E)$ is open in $K(X)$. A linear (i.e. additive and homogeneous) mapping $K: X \to Y$ is relatively open $\Longleftrightarrow$ if there exists a constant $M > 0$ so that for each $y \in K(X)$ there exists $x \in X$ such that

(1) $y = Kx$ and $\|x\| \leq M \|y\|$.

A mapping $G: B_r(0) \to Y$, $B_r(0) \subset X$, is said to be closed on $B_r(0)$ if $u_n, u \in B_r(0)$,

$u_n \to u$, $G(u_n) \to v \Rightarrow v = G(u)$. The following theorem generalizes the results of Graves [5] and Bartle [6].

Theorem 4. Let X be a Banach space, Y a normed linear space, $G : B_r(0) \to Y$ a closed mapping of $B_r(0) \subset X$ into Y so that $G(0) = 0$. Assume $K : X \to Y$ is linear and relatively open so that $\|G(u) - G(v) - K(u-v)\| \leq \alpha \|u - v\|$, $(\alpha > 0)$,

for each $u, v \in B_r(0)$, where $\alpha M < 1$ (here M is a constant from (1)). Then the equation $G(u) = y$ has a solution u in $B_r(0)$ provided $\|y\| < \varrho = r(1 - M\alpha)/M$.

In comparison with Graves and Bartle we need not assume that Y is complete, G, K are continuous (or G is Fréchet differentiable) and that K is onto Y. Theorems 3,4 are connected with openness of nonlinear operators.

Some other and rather general results concerning this matter will be published in [7].

The following theorems are related to those of Belluce-Kirk [8], Kirk [9] and Daneš [10].

Theorem 5. Let X be a normed linear space, $M \subset X$ a convex subset of X containing 0, $F : M \to M$ a mapping so that $u, v \in M$, $u \neq v \Rightarrow \|F(u) - F(v)\| < \|u - v\|$. Assume that for some $c > 0$ the set $\{u \in M : \|u - F(u)\| \leq c\}$ is non-void and weakly compact and that $\varphi(u) = \|u - F(u)\|$ is quasi-convex on M. Then there exists a unique point $u^* \in M$ o that $F(u^*) = u^*$.

<u>Theorem 6</u>. Let X, Y be normed linear spaces, $M \subset X$ a convex open subset of X, $0 \in M$, $F: M \longrightarrow Y$ a map such that $E(c) = \{u \in M : \|F(u)\| \leq c\}$ is non-void and weakly compact for some $c > 0$. Suppose $G : X \longrightarrow Y$ is n-positively homogeneous mapping of X onto Y . Let for each point $u \in M$ there exist constants α_u, δ_u $(0 \leq \alpha_u < 1$, $\delta_u > 0)$ so that $B_{\delta_u}(u) \subset M$ and $v \in B_{\delta_u}(u) \Longrightarrow$ $\Longrightarrow \|F(v) - F(u) - G(v-u)\| \leq \alpha_u \|G(v-u)\|$. If either a) F is weakly continuous on M , or b) F is demicontinuous on M , and $\psi(u) = \|F(u)\|$ is quasi-convex on M , then there exists $u^* \in M$ so that $F(u^*) = 0$.

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