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A SHRINKING OF A CATEGORY OF SOCIETIES IS A UNIVERSAL PARTLY ORDERED CLASS

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Introduction and summary. Partly ordered class $(P, \leqq)$ (i.e. a class P together with a reflexive and transitive binary relation on P) is called universal if every partly ordered class can be isomorphically embedded into $(P, \leqq)$.

All partly ordered classes can be considered as shrinking of categories:

If K is a category then a shrinking of K is a class of objects of K together with a partly ordering $\leqq$ defined by $a \leqq b$ if and only if there is a morphism of a from b into b .

In [1] it is proved that, under an assumption of non-existence of measurable cardinals, the shrinkings of binding categories are universal. A binding category is e.g. the category of all algebras with n-ary operations, $n \geqq 2$, and their homomorphisms. For the definition of a binding category and the other examples see [1].

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The main result of this paper is

Theorem 1. In the Gödel-Bernays set theory, the shrinking of tne category of societies and their compatible mappings ([2]) is a universal partly ordered class.

(A society is a couple (X, P) where X is a set and P is a family of non-empty subsets of X. A compatible mapping from (X, P) into (Y, R) is a mapping $f: X \longrightarrow Y$ such that $f(U) \in R$ for every $U \in P$.)

The proof of the theorem 1 is based upon the theorem 1 of [1] which says that the shrinking of the category Inc (see below) is universal:

Objects of Inc are indexed families of sets $(A_i, i \in I)$, A_i, I sets,

morphisms of Inc from $(A_i, i \in I)$ into $(B_j, j \in J)$ are all mappings $f: I \longrightarrow J$ such that $A_i \supset B_{f(i)}$ for every $i \in I$,

a composition of morphisms is a composition of mappings.

The theorem 1 is an easy consequence of

Theorem 2. There is a full embedding from Inc into the category of societies.

(A full embedding is a one-to-one functor $F: K \longrightarrow L$ which maps K onto a full subcategory of L.)

The proof of the theorem 2 is divided into three steps:

1) A full embedding of the category of all sets and

identities into the category of societies (§ 2).

2) A full embedding of the dual of the category of all sets and inclusions into the category of societies (§ 3).

3) A full embedding of the category $\mathcal{I}nc$ into the category of societies (§ 4).

In the paragraph 1 we shall prove some lemmas. As a consequence of the theorem 1, we shall construct a simple universal concrete category (see [3]) from binary relations and societies in the paragraph 5.

§ 1. <u>Definition</u>. A category $\mathcal{S}oc\,(m)$, m natural, is defined as follows:

Objects of $\mathcal{S}oc\,(m)$ are $m+1$ -tuples $(X, P_1, \dots, P_m)$, where X is a set and $P_1, \dots, P_m$ are families of nonempty subsets of X ,

morphisms of $\mathcal{S}oc\,(m)$ from $(X, P_1, \dots, P_m)$ into $(Y, R_1, \dots, R_m)$ are all mappings $f : X \to Y$ such that $f(U) \in R_i$ for every $i = 1, \dots, m$ and $U \in P_i$,

a composition of morphisms is a composition of mappings.

$\mathcal{S}oc\,(1)$ (the category of societies) will be denoted by $\mathcal{S}oc$.

<u>Lemma 1</u>. Given a natural m , there exists a full embedding $\mathcal{S}oc\,(m) \longrightarrow \mathcal{S}oc$.

<u>Proof</u>. It is proved in [2] that there is a connected rigid 2-society (Z, S) (i.e. if x, y are points of Z then there is a sequence $U_0, \dots, U_k$ of elements of S such that $x \in U_0$, $y \in U_k$ and $U_{i-1} \cap \cap U_i \neq \emptyset$ for $i = 1, \dots, k$; only compatible

mapping of $(Z, \mathcal{S})$ into itself is the identity; elements of $\mathcal{S}$ are two-point sets) such that Z has at least $3n$ points. We can suppose without loss of generality that $Z = \{1, 2, 3, \dots, m\}$, where $m \geq 3n$.

A full embedding $F : \mathcal{Soc}(n) \longrightarrow \mathcal{Soc}$ is defined as follows:

$$F((X, P_1, \dots, P_n)) = (X \times Z, A_0 \cup A_1 \cup \dots \cup A_n),$$

where $U \in A_0$ if and only if there is $x \in X$ and $V \in \mathcal{S}$ such that $U = \{x\} \times V$,

$U \in A_i$ if and only if there is $V \in P_i$ such that $U = V \times \{3i-2, 3i-1, 3i\}$ for $i = 1, \dots, n$,

$F(f) = f \times id_Z$.

It is evident that F is a one-to-one functor from $\mathcal{Soc}(n)$ into $\mathcal{Soc}$. We shall prove that F maps $\mathcal{Soc}(n)$ onto a full subcategory of $\mathcal{Soc}$:

Let $M = (X, P_1, \dots, P_n)$ and $N = (Y, R_1, \dots, R_n)$ be objects of $\mathcal{Soc}(n)$ and f be a compatible mapping from $F(M) = (X \times Z, A_0 \cup \dots \cup A_n)$ into $F(N) = (Y \times Z, B_0 \cup \dots \cup B_n)$.

Elements of A_0 have two points, elements of $B_1, \dots, B_n$ have at least three points. Therefore f maps elements of A_0 onto elements of B_0.

If $i \in Z$ then $1, i$ are connected by a chain of elements of $\mathcal{S}$. Therefore if $x \in X$ then $(x,1)$, (x,i) are connected by a chain of elements of A_0, which implies that $f((x,1))$, $f((x,i))$ are connected by a chain of elements of B_0.

According to a definition of B_0, the first coordinates of both $f((x,1))$ and $f((x,i))$ are the same. Hence there are mappings $g: X \to Y$ and $h_x: Z \to Z$, $x \in X$ such that $f((x,i)) = (g(x), h_x(i))$ for every $x \in X$, $i = 1, \dots, m$.

Let x be an element of X. Then $h_x: Z \to Z$ is a compatible mapping from (Z, S) into itself, because if $U \in S$ then
$\{x\} \times U \in A_0$, $f(\{x\} \times U) = \{g(x)\} \times h_x(U) \in B_0$,
which implies $h_x(U) \in S$.

As (Z, S) is a rigid society, all h_x are the identities, which implies that $f = g \times id_Z$.

If i is a natural number less than m and $U \in P_i$ then $U \times \{3i-2, 3i-1, 3i\} \in A_i$. Therefore $f(U \times \{3i-2, 3i-1, 3i\}) = g(U) \times \{3i-2, 3i-1, 3i\} \in B_i$, which implies $g(U) \in R_i$.

Hence a mapping g is a compatible mapping from M into N and $f = F(g)$.

Thus, we have proved that F is a full embedding.

The next lemma enables us to simplify the proofs of the theorems 2,4.

<u>Lemma 2</u>. There exists a full embedding of $\mathcal{Soc}$ into itself such that for every different objects M, N of $\mathcal{Soc}$ the underlying sets of $F(M)$ and $F(N)$ are disjoint and do not contain $\emptyset$ as an element.

<u>Proof</u>. A full embedding $F: \mathcal{Soc} \to \mathcal{Soc}$ is defined as follows:
If $M = (X, P)$ is a society then
$F(M) = (X \times \{M\}, P')$, where $U \in P'$ if and only if there is $V \in P$ such that $U = V \times \{M\}$,

if f is a compatible mapping from M into N then $F(f)((x, M)) = (f(x), N)$.

The details are left to the reader.

§ 2. <u>Theorem 3</u>. There is a full embedding of the category of all sets and identities into Soc .

<u>Proof</u>. It follows from the axiom of choice and the lemma 1 that it is sufficient to construct a full embedding F from the category of all ordinals greater than 1 and identities on them into $Soc(3)$.

A set of ordinals less than n will be denoted by L_n .

A full embedding F is defined by

$F(n) = (L_n, 2(n), \{L_n\}, \{L_k : k \leq n\})$,

where $2(n)$ is a family of all two-point sets of cardinals less than n , $F(id_n) = (id_{F(n)})$.

It is evident that F is a one-to-one functor.

Let m, n be ordinals and f be a compatible mapping from $F(m)$ into $F(n)$.

f is a one-to-one mapping, because if $p < q < m$ then $\{p, q\} \in 2(m)$, $f(\{p, q\}) \in 2(n)$, which implies $f(p) \neq f(q)$.

f maps L_m onto L_n , since $L_m \in \{L_m\}$, which implies $f(L_m) \in \{L_n\}$, $f(L_m) = L_n$.

f is monotone, because if $p < q < m$ and $f(q) \leq f(p)$ then there is $r < n$ such that $L_r = f(L_q)$ (see $L_q \in \{L_k : k \leq m\}$). Therefore $f(q) \in f(L_q)$ and there is $s < q$ such that $f(q) = f(s)$, which is a contradiction.

As f is a monotone-1-1 mapping of the well-ordered set L_m onto the well-ordered set L_n, it is $m = n$ and f is the identity.

Thus, we have proved that F is a full embedding.

§ 3. Theorem 4. There exists a full embedding of the dual of the category of all sets and inclusions into $\mathcal{Soc}$.

Proof. Let F be a full embedding of the category of sets and identities into $\mathcal{Soc}$ (Theorem 3).

Denote $F(X)$ by (S_X, R_X). According to the lemma 2, we can suppose that $\emptyset \notin S_X$, $S_X \cap S_Y = \emptyset$ for every different sets X, Y.

It is sufficient to construct a full embedding G from the dual of the category of sets and inclusions into $\mathcal{Soc}(3)$:

$$G(A) = (\{\emptyset\} \cup \bigcup_{x \in A} S_x,$$
$$\{\{\emptyset\}\}, \{\{\emptyset\} \cup S_x, x \in A\} \cup \{\{\emptyset\}\},$$
$$\bigcup_{x \in A} R_x, \{\{\emptyset\} \cup \bigcup_{x \in A} S_x\}),$$

$$G(A \supset B)(u) = \begin{cases} u & \text{if there is } x \in B \text{ such that } u \in S_x, \\ \emptyset & \text{if } u \in S_x \text{ for no } x \in B. \end{cases}$$

$f = G(A \supset B)$ is a compatible mapping from $F(A)$ into $F(B)$, because $f(\emptyset) = \emptyset$, f maps S_x, $x \in A$ either identically onto S_x or onto $\emptyset$ and maps the underlying set of $G(A)$ onto the underlying set of $G(B)$.

Thus, G is a one-to-one functor.

Let A, B be sets and f be a compatible mapping from $G(A)$ into $G(B)$.

It is obvious that $f(\emptyset) = \emptyset$. If $z \in A$ then $\{\emptyset\} \cup S_z \in \{\{\emptyset\} \cup S_x, x \in A\} \cup \{\{\emptyset\}\}$.
Hence either $f(\{\emptyset\} \cup S_z) = \{\emptyset\} \cup S_w$, $w \in B$ or $f(\{\emptyset\} \cup S_z) = \{\emptyset\}$. In the first case, a restriction of f to S_z is a compatible mapping from (S_z, R_z) into (S_w, R_w) and from the properties of F it follows that $z = w$, the restriction of f to S_z is the identity.

If $u \in S_z$, $z \in A - B$ then u is not an element of an underlying set of $G(B)$. Therefore it must be $F(u) = \emptyset$.

It is obvious that f is onto $\{\emptyset\} \cup \bigcup_{x \in B} S_x$. Hence if $u \in S_z$, $z \in B$ then there is $v \in \{\emptyset\} \cup \bigcup_{x \in A} S_x$ such that $f(v) = u$. It is obvious that $u = v$. Hence it is $f(u) = u$, $z \in A$.

We have proved that $A \supset B$ and $f = G(A \supset B)$.

§ 4. <u>Proof of the theorem 2</u>. Let G be a full embedding of the dual of the category of sets and inclusions into $\mathcal{S}oc$ (Theorem 4).

Denote $G(A) = (T_A, P_A)$, $G(A \supset B) = \varphi_{A,B}$. According to the lemma 2, we can suppose that $T_A \cap T_B = \emptyset$ for every different sets A, B.

It is sufficient to construct a full embedding H from $\mathcal{I}nc$ into $\mathcal{S}oc(2)$:

$$H((A_i, i \in I)) = (\bigcup_{i \in I} T_{A_i}, \{X : X \subset T_{A_i}, i \in I\}, \bigcup_{i \in I} P_A),$$

$$H(f)(u) = q_{A_i, B_{f(i)}}(u) \quad \text{for} \quad u \in T_{A_i} .$$

We can see that H is a one-to-one functor. Let $(A_i, i \in I) = M$ and $N = (B_j, j \in J)$ be objects of *Inc* and q be a compatible mapping from $H(M)$ into $H(N)$.

It is $T_{A_k} \in \{X : X \subset T_{A_i}, i \in I\}$, which implies $f(T_{A_k}) \in \{X : X \subset T_{B_j}, j \in J\}$. Therefore there is a mapping $f : I \longrightarrow J$ such that q maps T_{A_k} into $T_{B_{f(k)}}$ for $k \in I$.

Evidently, a restriction of q to T_{A_k} is a compatible mapping from (T_{A_k}, P_{A_k}) into $(T_{B_{f(k)}}, P_{B_{f(k)}})$. Therefore $A_k \supset B_{f(k)}$ and $q(u) = q_{A_k, B_{f(k)}}(u)$ for $u \in T_{A_k}$, $k \in I$.

We have proved that H is a full embedding of *Inc* into $\mathcal{S}oc\,(2)$.

§ 5. A concrete category is a couple $(\mathcal{K}, F)$, where $\mathcal{K}$ is a category and F is a faithful functor from $\mathcal{K}$ into the category of sets and mappings.

A concrete category $(\mathcal{K}, F)$ is called universal if for every concrete category $(\mathcal{L}, G)$ there exists a full embedding $H : \mathcal{L} \longrightarrow \mathcal{K}$ with $G = FH$.

Define a concrete category $(\mathcal{U}, E)$ as follows: objects of $\mathcal{U}$ are couples $(X, (A_R, R \subset X \times X))$,

where A_R are societies;
morphisms of $\mathcal{U}$ from $(X, (A_R, R \subset X \times X))$ into $(Y, (B_S, S \subset Y \times Y))$ are all mappings $f: X \to Y$ such that if R is a binary relation on X and $S = \{(f(x), f(y)): (x, y) \in R\}$ is a relation on Y then there is a compatible mapping from A_R into B_S,
a composition of morphisms is a composition of the corresponding mappings,
an underlying set of $(X, (A_R, R \subset X \times X))$ is X,
an underlying mapping of a morphism f is f itself.

As a corollary of the theorem 1 we have the next theorem:

Theorem 5. The concrete category $(\mathcal{U}, E)$ is universal.

The proof of the theorem 5 can be obtained from the proof of the Theorem of [3] if we replace binary algebras by societies and homomorphisms by compatible mappings. Instead of Theorem 1 of [1] we must use Theorem 1 of the present paper.

References

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