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A REMARK TO THE FINITE ELEMENT METHOD

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I. Introduction. The finite element method, its different versions has become very important today in the theory and practice. See e.g. [1] - [17] and others.

One important form of the method is such that the approximate solution $u_h(X)$ on the domain Ω has the following form

(1.1) $$u_h(\underline{X}) = \sum c(h, \underline{k})\, \varphi\left(\frac{\underline{x}}{h} - \underline{k}\right)$$

where the function $\varphi(X)$ has a compact support and the sum is over all multiintegers $\underline{k} \equiv (k_1, \ldots, k_m)$, ($k_i$ integers) such that

$$\operatorname{supp} \varphi\left(\frac{\underline{x}}{h} - \underline{k}\right) \cap \Omega \neq \emptyset \ .$$

Many times the function $\varphi(x)$, $\underline{x} \in R_m$ is taken in a special form

(1.2) $$\varphi(X) = \prod_{i=1}^{m} \varphi(x_i), \quad i = 1, 2, \ldots, m, \; X \equiv (x_1, \ldots, x_m).$$

For some domains e.g. a square, for $\varphi(X)$ smooth enough so called "inverse theorem" is valid for some sequences h_i, $i = 1, 2, \ldots$, $h_i \to 0$.

(1.3) $$\| u_{h_i}(\underline{x}) \|_{W_2^{\ell}(\Omega)} \leq C(\ell)\, h_i^{-\ell}\, \| u_{h_i} \|_{W_2^{0}(\Omega)} \ .$$

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where $W_2^{\ell}(\Omega)$ is the usual Sobolev space and $C(\ell)$ does not depend on i.

Many very important theorems of the theory of the finite elements method are based on the validity of the mentioned "inverse theorem". Under this assumption e.g. the computation of derivatives (see [14]) could be made much easier as in [14], simply by derivating the approximate solution. Without this "inverse theorem" proofs of many theorems will break down.

The purpose of this note is to show that for smooth domains the inverse theorem is not true for any choice of h_i.

To simplify the arguments we shall deal only with the two dimensional case i.e. $\Omega \subset R_2$.

The inverse theorem is in fact equivalent with a geometric property of the domain Ω. Let us explain this property.

Let $\Omega \subset R_2$ be a bounded domain.

For every $h > 0$ let us define the mesh $P_h = \{(hk, h\ell)\}$ with k, ℓ integers.

Further let Q_h be the set of the all squares $S^h_{k,\ell}$

$$S^h_{k,\ell} \equiv \{x = (x_1, x_2);\ hk < x_1 < h(k+1),\ h\ell < x_2 < h(\ell+1)\}$$

and

$$Q^{\Omega}_h \equiv \{\bigcup_{k,\ell} S^h_{k,\ell};\ S^h_{k,\ell} \cap \Omega \neq \emptyset\},$$

$$P^{\Omega}_h = \{(k,\ell);\ S^h_{k,\ell} \in Q^{\Omega}_h\}.$$

Let us define now the function $\psi_h(k,\ell)$ on P_h by the following way:

$$(1.4) \qquad \psi_h(k,l) = \frac{mes\,(S^h_{k,l} \cap \Omega)}{h^2}$$

where $(S^h_{k,l} \cap \Omega)$ means the measure of the set $S^h_{k,l} \cap \Omega$. Further let

$$(1.5) \qquad \chi(h) = \min_{k,l} \psi_h(k,l), \quad (k,l) \in P^\Omega_h .$$

The crucial question is whether we may choose a sequence $h_j \to 0$, $j = 1, 2, \ldots$ such that

$$(1.6) \qquad \lim_{j\to\infty} \inf \chi(h_j) \geq \alpha .$$

Certainly if Ω is a square then such a sequence h_j clearly exists.

In the next chapter we shall show that such a sequence does not exist in the case that the boundary has bounded curvature.

For the simplicity we shall prove the statement for a circular domain only. By the same idea it is obviously possible to prove the general statement for domains with bounded curvature in R_m.

II. <u>Theorem</u>. As we said we shall study the case of the circular domain Ω in R_2 only. Let

$$(2.1) \qquad K(r) \equiv \{(x,y);\ x^2 + y^2 < r^2\}$$

be a circle with the radius r.

Further let us denote $\rho(x,y)$ the distance of the point (x,y) to the boundary of the circle $K(r)$.

Let us prove the following theorem.

Theorem. Let us denote

(2.2) $$\varkappa(\kappa) = \min_{(k,l)\in K(\kappa)} \rho(k,l) \quad ,$$

k, l integers.

Then for every sequence $\kappa_i \to \infty$ we have

(2.3) $$\liminf_{i\to\infty} \varkappa(\kappa_i) = 0 \quad .$$

This theorem is obviously equivalent with the statement which we introduced in the first chapter in the case that Ω is a circle.

Proof. 1) Let us denote by $U(\kappa)$ the number of all (k, l) in $K(\kappa)$,

(2.4) $$U(\kappa) = \sum_{(k,l)\in K(\kappa)} 1 \quad .$$

Obviously $U(\kappa)$ is a not decreasing step function. Let ξ_i , $\xi_{i+1} > \xi_i$ be the sequence of all points of the discontinuity of the function $U(\kappa)$.

Denote further

(2.5) $$\lambda(\kappa) = \sup_{\xi_i \geq \kappa} |\xi_i - \xi_{i-1}| \quad .$$

Our theorem will be proved if we shall show that

(2.6) $$\lim_{\kappa\to\infty} \lambda(\kappa) = 0 \quad .$$

In fact

(2.7) $$\varkappa(\kappa_i) \leq \lambda(\kappa_i) \quad .$$

2) Let us assume on the contrary that there exists a sequence κ_i , $i = 1, 2, \dots$, $\kappa_{i+1} > \kappa_i$, $\kappa_i \to \infty$ such that

(2.8) $$\liminf_{i \to \infty} \varkappa(\kappa_i) \geq 3\Delta > 0 .$$

Then for all κ_i, $i \geq N$ we have

(2.9) $$\lambda(\kappa_i) \geq 2\Delta .$$

So we may construct the subsequence ξ_{i_j},
such that

(2.10) $$\xi_{i_j} - \xi_{i_j - 1} \geq \Delta$$

and

$$\xi_{i_j} \geq \kappa_j .$$

Let us denote

(2.11) $$\omega_j = [\xi_{i_j - 1}]$$

where $[x]$ means integral part of x.

We have

(2.12) $$\omega_j \leq \xi_{i_j - 1} \leq \omega_j + 1 .$$

Because ω_j and $\omega_j + 1$ is certainly the point of the discontinuity of the function $U(\kappa)$ we have

(2.12) $$\omega_j \leq \xi_{i_j - 1} \leq \xi_{i_j - 1} + \Delta \leq \xi_{i_j} \leq \omega_j + 1$$

and

(2.13) $$\xi_{i_j - 1} = \omega_j + \sigma_j \quad \text{where} \quad 0 \leq \sigma_j \leq 1 - \Delta .$$

Let us define now the numbers α_j resp. β_j such that

(2.14) $$\omega_j^2 + \alpha_j^2 = \xi_{i_j - 1}^2 = (\omega_j + \sigma_j)^2 ,$$

(2.15) $$\omega_j^2 + \beta_j^2 = \xi_{i_j}^2 \geq (\omega_j + \sigma_j + \Delta)^2 .$$

Therefore

$$(2.16) \quad \alpha_j = \sqrt{(\omega_j + \sigma_j)^2 - \omega_j^2} = \omega_j^{\frac{1}{2}} \sigma_j^{\frac{1}{2}} (2 + \sigma_j \omega_j^{-1})^{\frac{1}{2}} ,$$

$$(2.17) \quad \beta_j \geq \omega_j^{\frac{1}{2}} (\sigma_j + \Delta)^{\frac{1}{2}} \; (2 + (\sigma_j + \Delta)\, \omega_j^{-1})^{\frac{1}{2}} \; .$$

But

$$\omega_j \geq r_j - 1 , \quad 0 \leq \sigma_j < 1 \; .$$

Therefore

$$(2.18) \qquad \alpha_j \leq \omega_j^{\frac{1}{2}} \sigma_j^{\frac{1}{2}} 2^{\frac{1}{2}} (1 + \frac{1}{(r_j - 1)2})^{\frac{1}{2}}$$

and hence

$$(2.19) \; \beta_j - \alpha_j \geq 2^{\frac{1}{2}} \omega_j^{\frac{1}{2}} [(\sigma_j + \Delta)^{\frac{1}{2}} - \sigma_j^{\frac{1}{2}} (1 + \frac{1}{(r_j - 1)2})^{\frac{1}{2}}] \; .$$

But

$$(2.20) \qquad \min_{0 \leq x \leq 1-\Delta} [(x + \Delta)^{\frac{1}{2}} - x^{\frac{1}{2}} (1 + \frac{1}{(r_j - 1)2})^{\frac{1}{2}}] =$$

$$= 1 - (1 - \Delta)^{\frac{1}{2}} (\underline{1} + \frac{1}{(r_j - 1)2})^{\frac{1}{2}} \; .$$

Therefore for r_j big enough i.e. $j > N_1$ we have

$$(2.21) \qquad \beta_j - \alpha_j \geq 2^{\frac{1}{2}} (r_j - 1)^{\frac{1}{2}} C \qquad \text{with } C > 0$$

and hence for $j \cdot N_2 > N_1$ we have $\beta_j - \alpha_j \geq 2$

and therefore there exists an integer k_j such that

$\beta_j > k_j > \alpha_j$ for all $j > N_2$

and hence $\varsigma_j = (\omega_j^2 + k_j^2)^{\frac{1}{2}}$ is a point of discontinuity of the function $\mathfrak{U}(r)$.

But obviously

$$\xi_{i_j-1} < \xi_j < \xi_{i_j-1} + \Delta$$

and this is a contradiction and the theorem is proved.

A closer analysis of the proof shows that obviously by the same idea the general statement introduced in § 1 may be proved i.e. for m -dimensional domain with bounded curvature of the boundary.

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